Sums-of-squares and module lattice isomorphisms

Alexandre Wallet, PQ Shield

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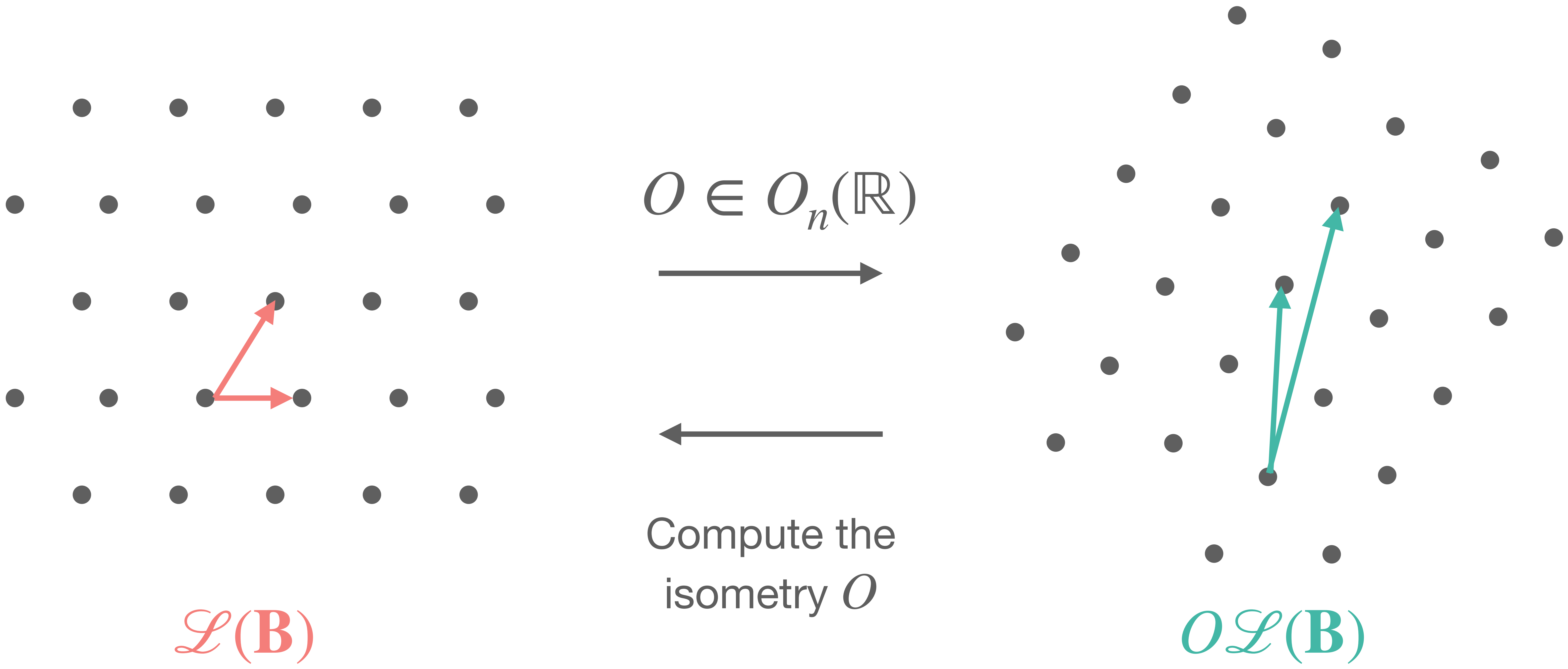
Based on joint works with C. Chevignard, T. Espitau, G. Mureau, A. Pellet—Mary, H. Pliatsok and P.A. Fouque

PQ SHIELD

The Lattice Isomorphism Problem (LIP)



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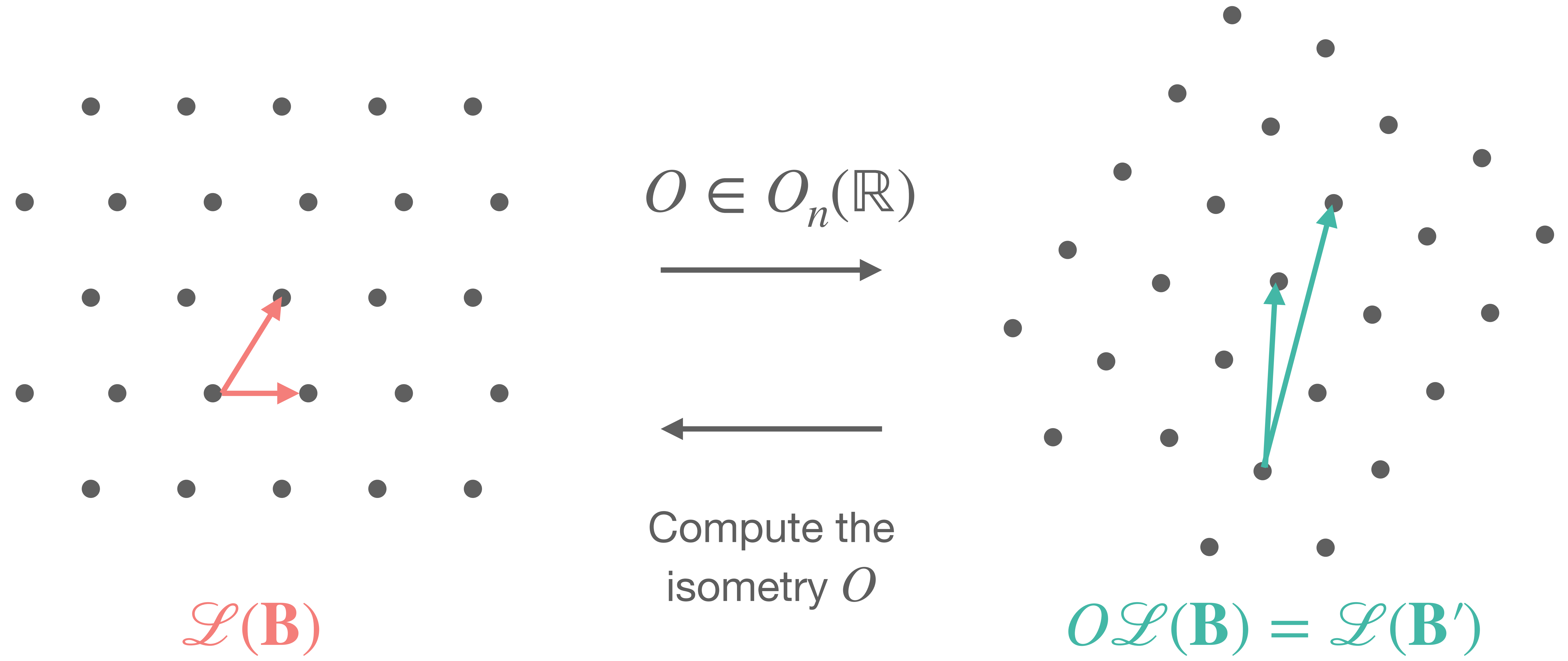
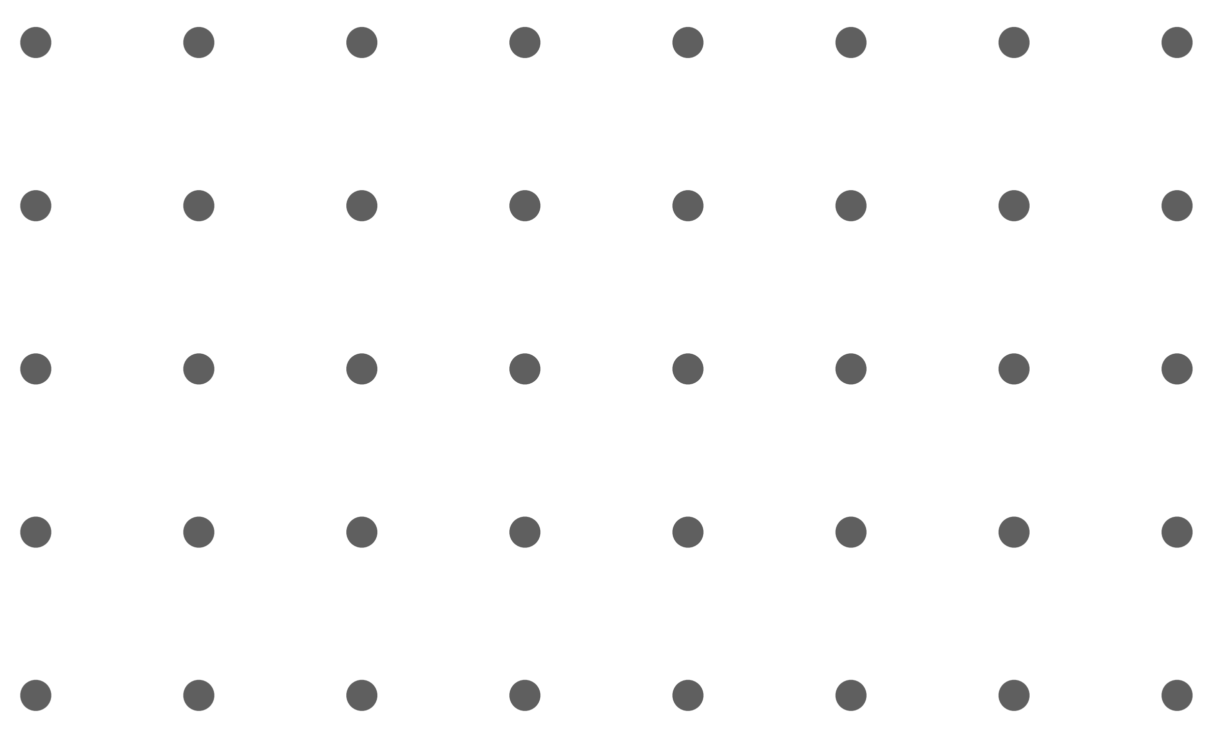


A computational version: Given $\mathbf{B}' = \mathbf{OBU}$, with $\mathbf{O}$ orthogonal and $\mathbf{U} \in \mathrm{GL}_n(\mathbb{Z})$, compute $\mathbf{O}$ or $\mathbf{U}$.

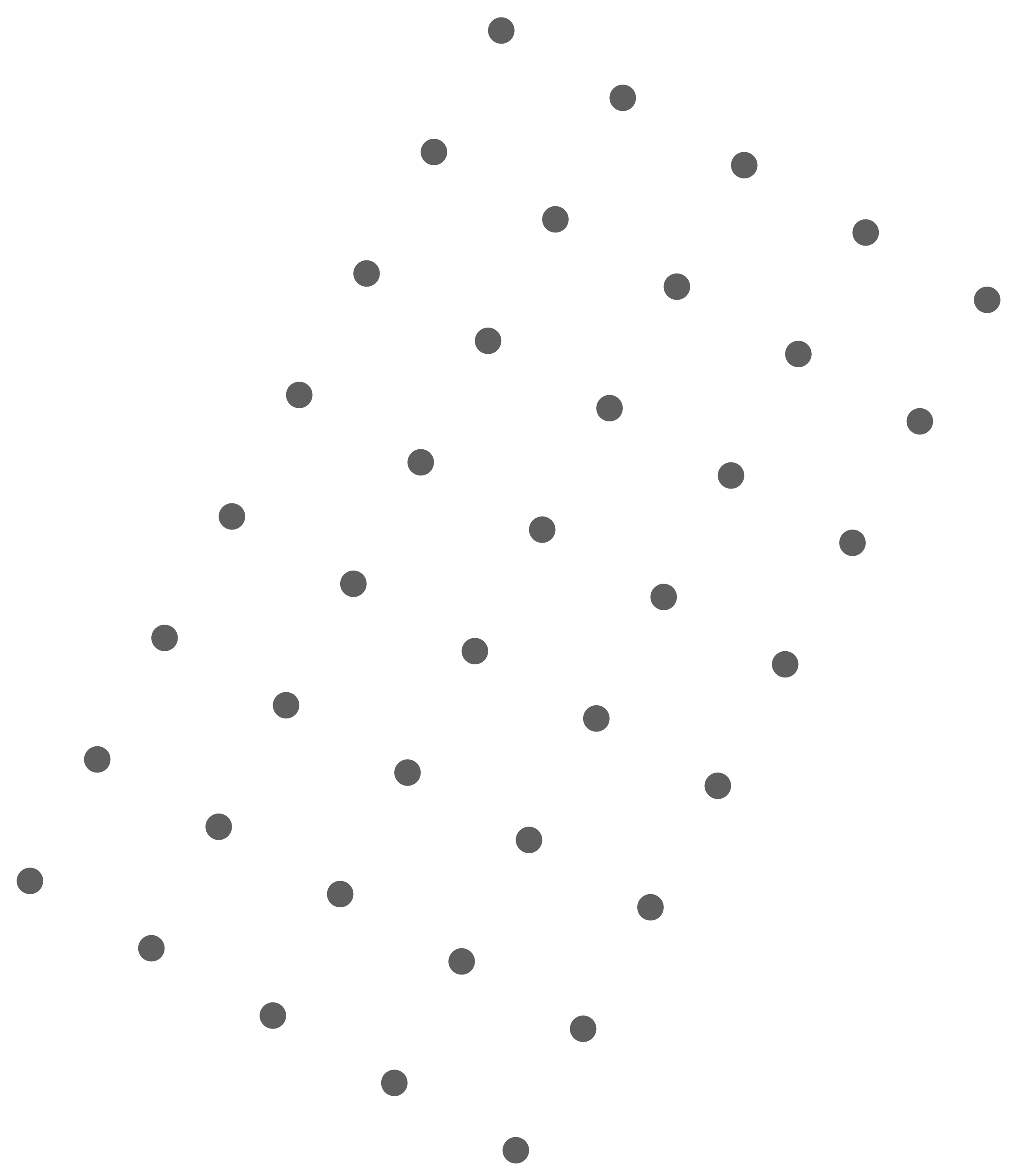
Proof of knowledge from LIP[1]

$\mathbb{Z}^n$



$\mathbf{G} = \mathbf{U}^t\mathbf{U}$

$\mathbf{OU}$



$O\mathbb{Z}^n$

$\mathbf{O}, \mathbf{U}$ secret

Prover

Verifier

[1] : L. Ducas and W. Van Woerden, e.g. ePrint 2021/1332

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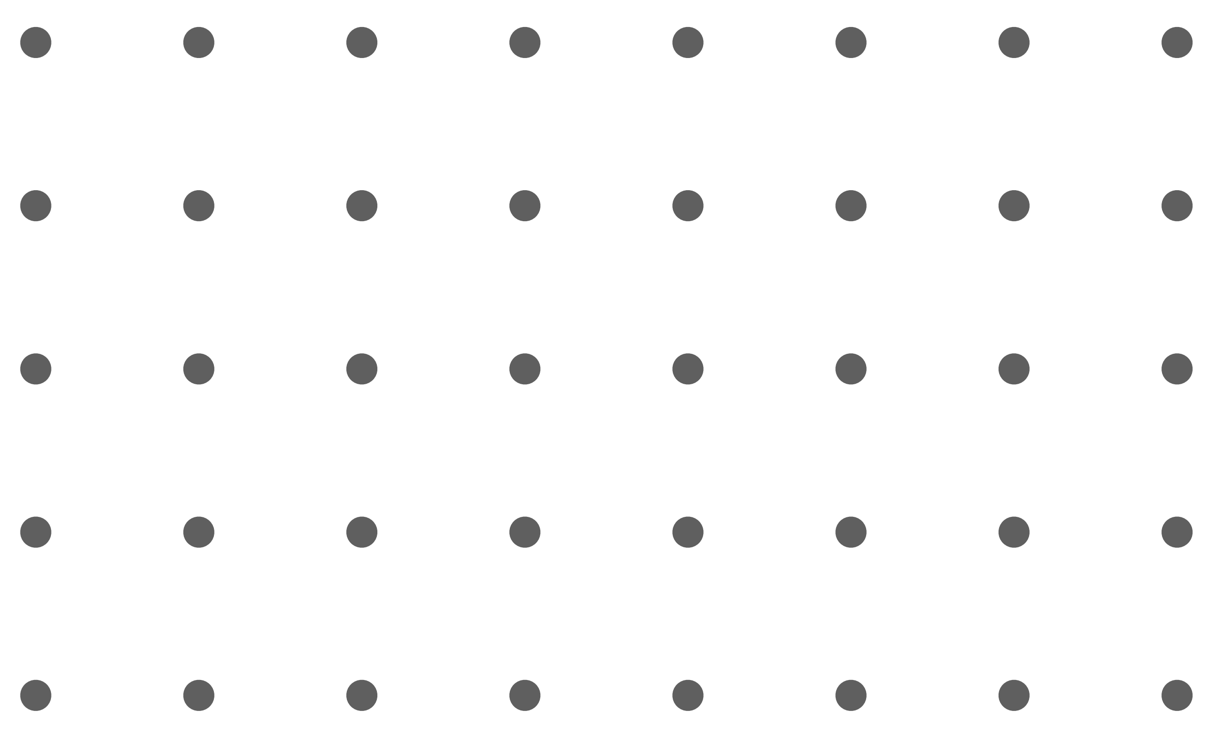

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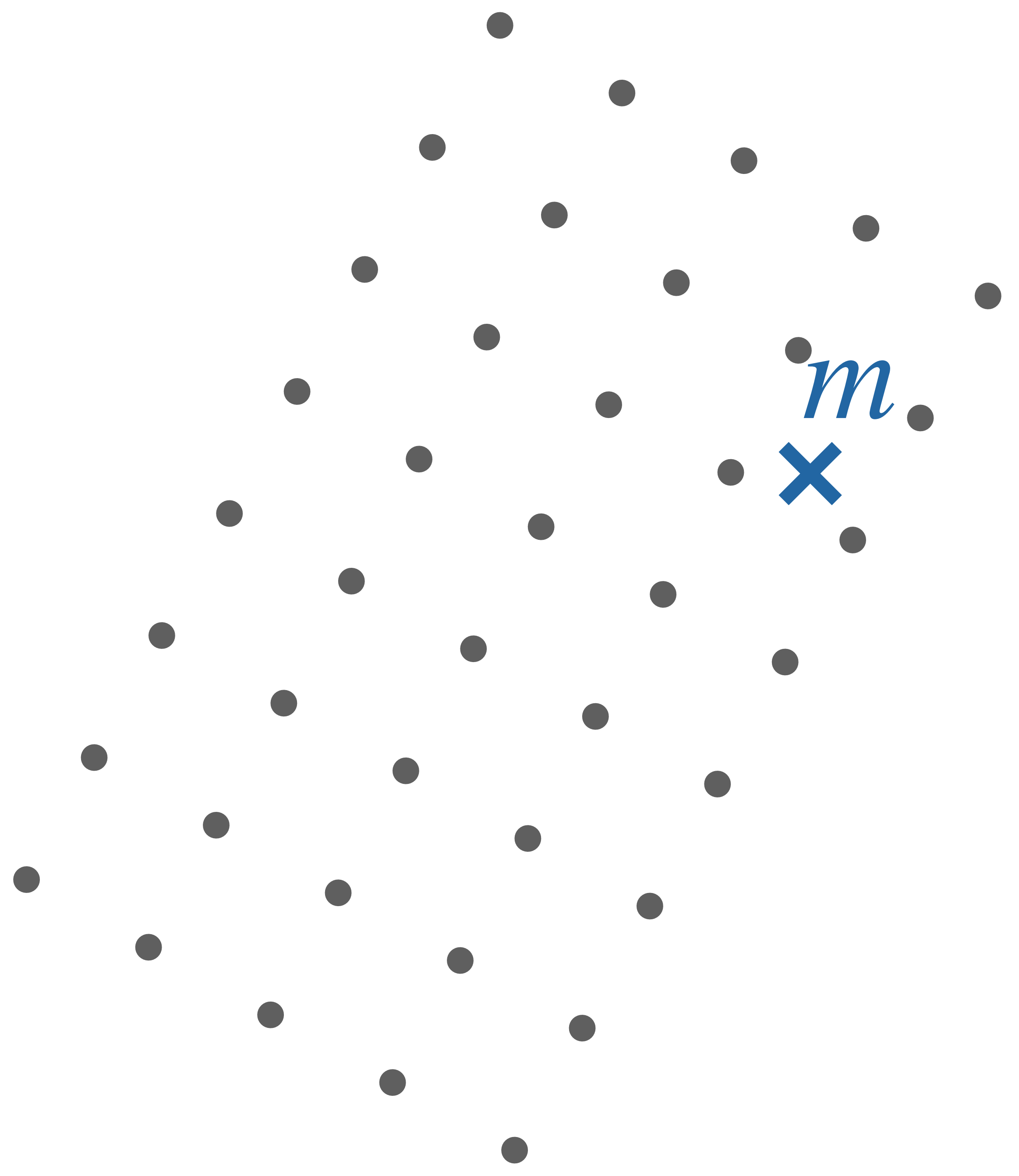

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$\tilde{m} = \mathbf{U}m$

m ←

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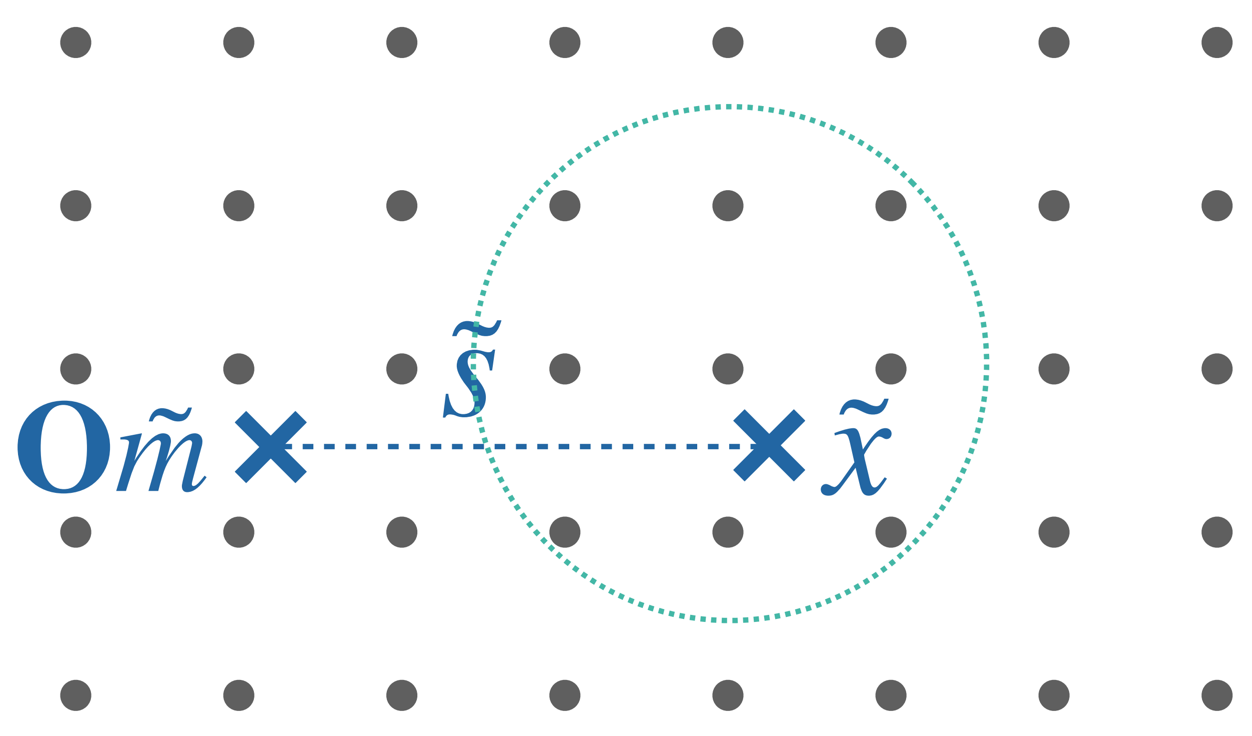

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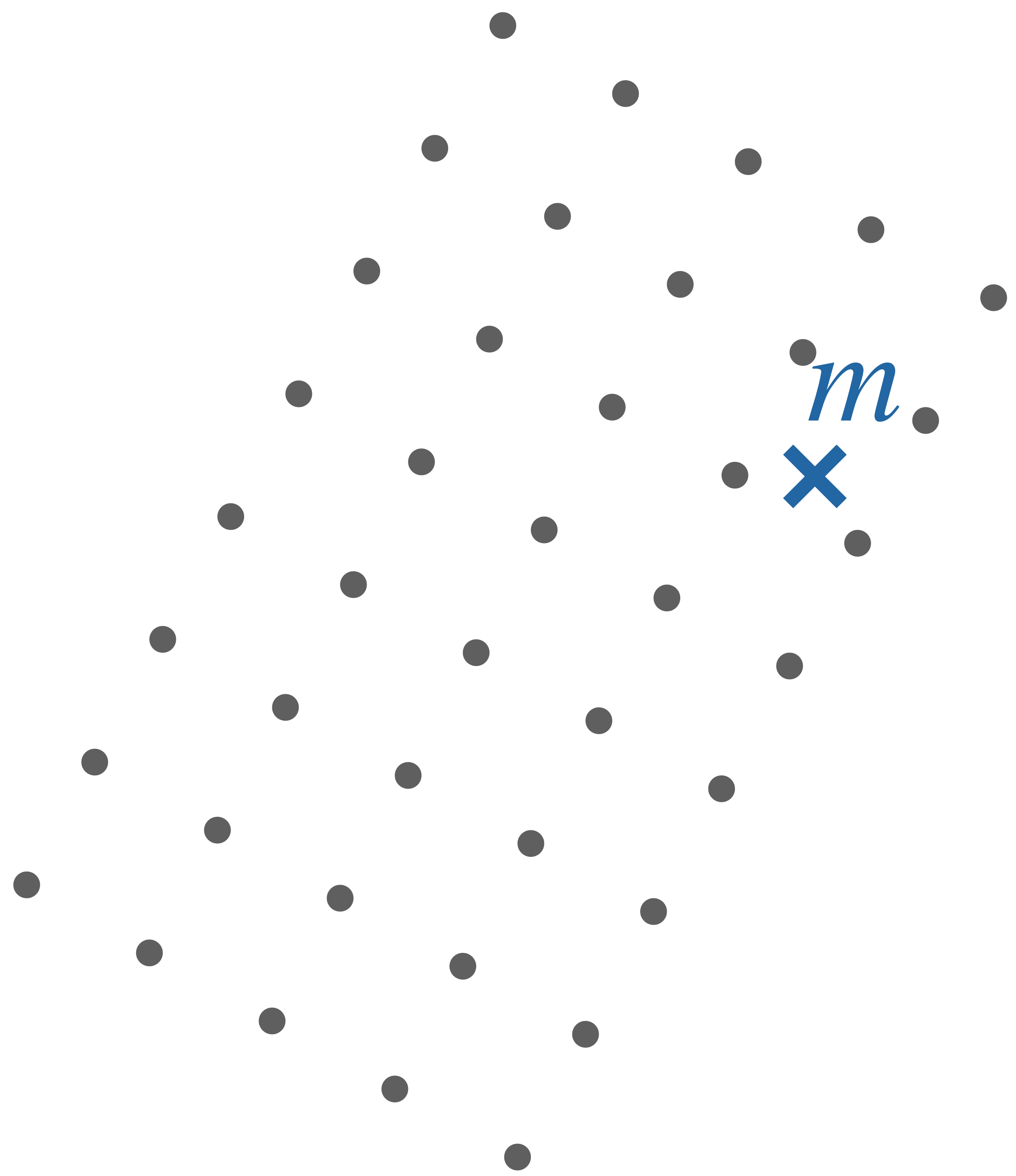

$O\mathbb{Z}^n$

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Prover

Verifier

$\tilde{m} = \mathbf{U}m \qquad m \longleftarrow$

Sample short $\tilde{x}$ in $\mathbf{O}\tilde{m} + \mathbb{Z}^n$

$$s = \mathbf{U}^{-1}\mathbf{O}^t\tilde{s} = m - \mathbf{U}^{-1}\mathbf{O}^t\tilde{x}$$

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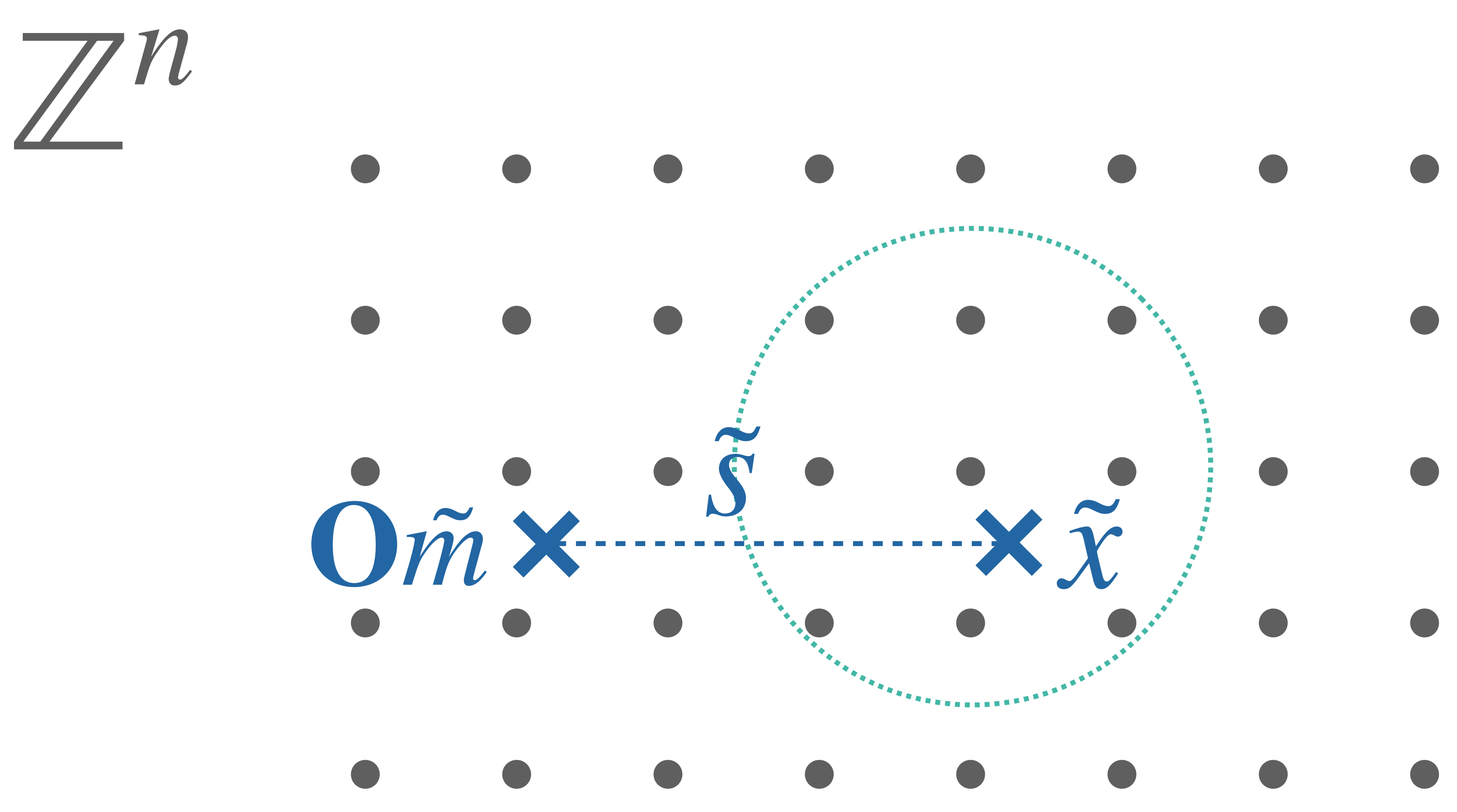
Proof of knowledge from LIP[1]



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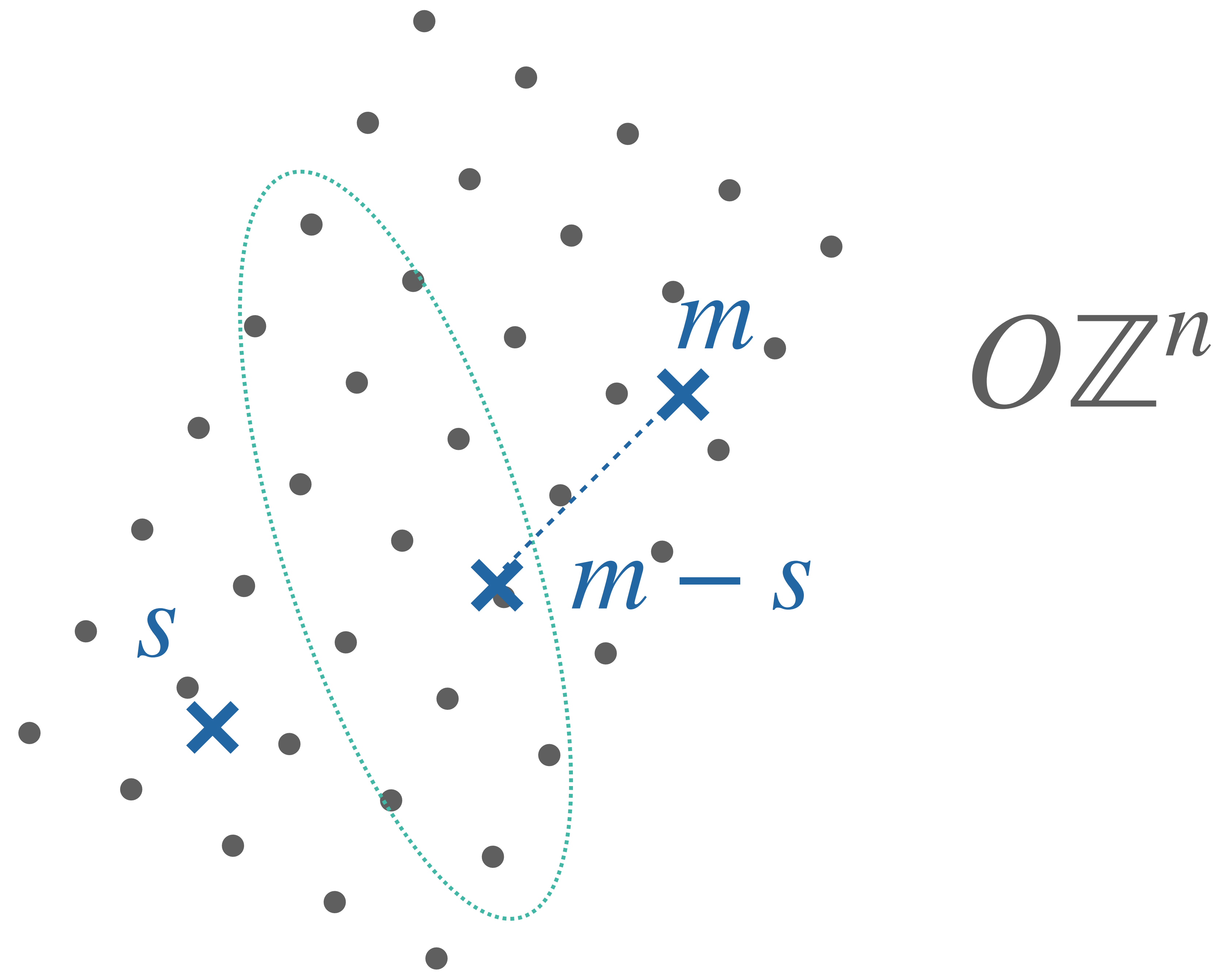

$\mathbf{O}, \mathbf{U}$ secret

Prover | Verifier

$\tilde{m} = \mathbf{U}m$ $\quad m \leftarrow$

Sample short $\tilde{x}$ in $\mathbf{O}\tilde{m} + \mathbb{Z}^n$

$s = \mathbf{U}^{-1}\mathbf{O}^t\tilde{s} = m - \mathbf{U}^{-1}\mathbf{O}^t\tilde{x}$ $\quad \rightarrow s$

1) $\mathbf{OU}s \in \mathbf{Z}^n$? 2) $(m-s)^t\mathbf{G}(m-s)$ is short? (where $(m-s)^t\mathbf{G}(m-s) = \|\tilde{x}\|^2$)

Yes both: the verifier is convinced!

(in the end, we do not care so much for O)

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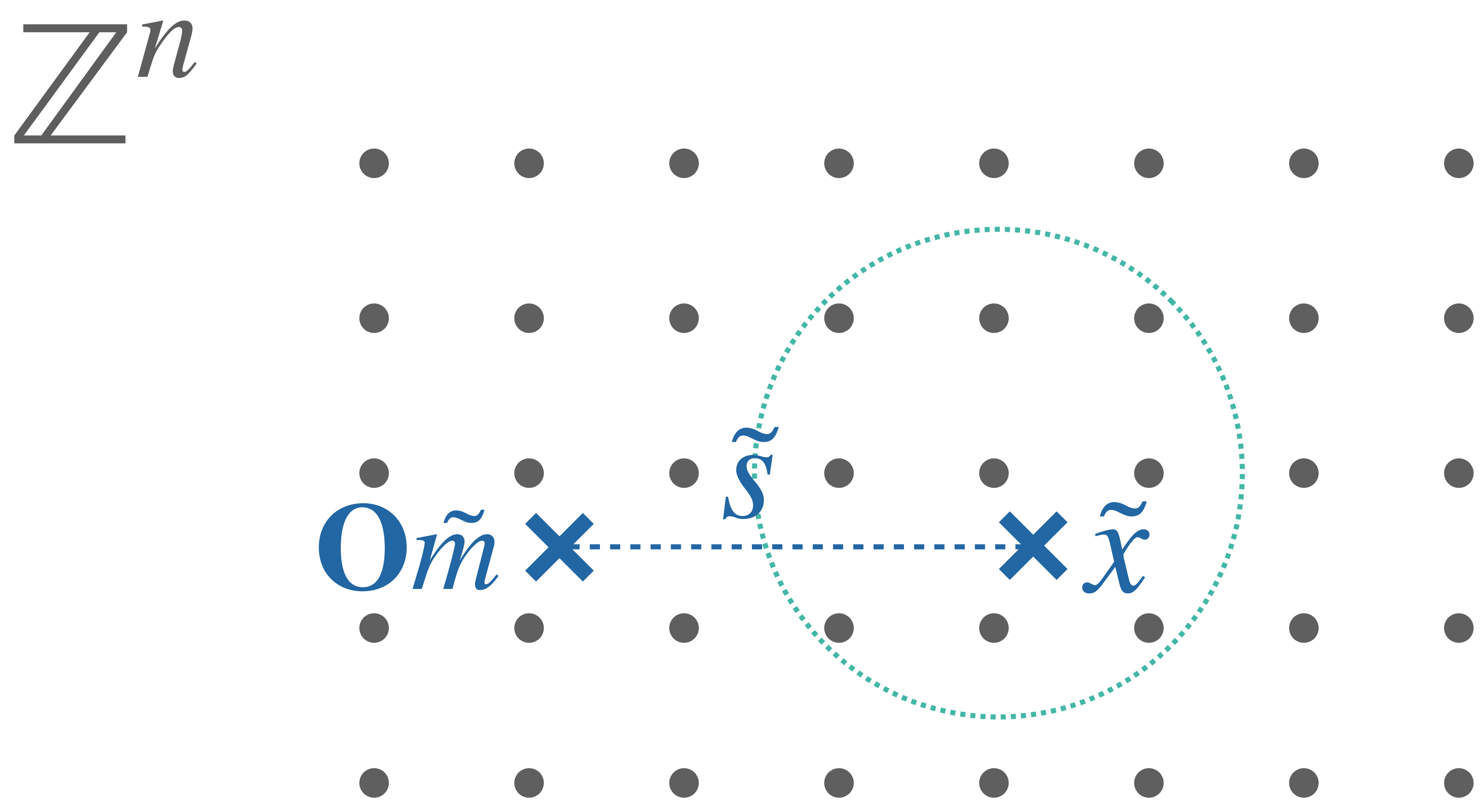

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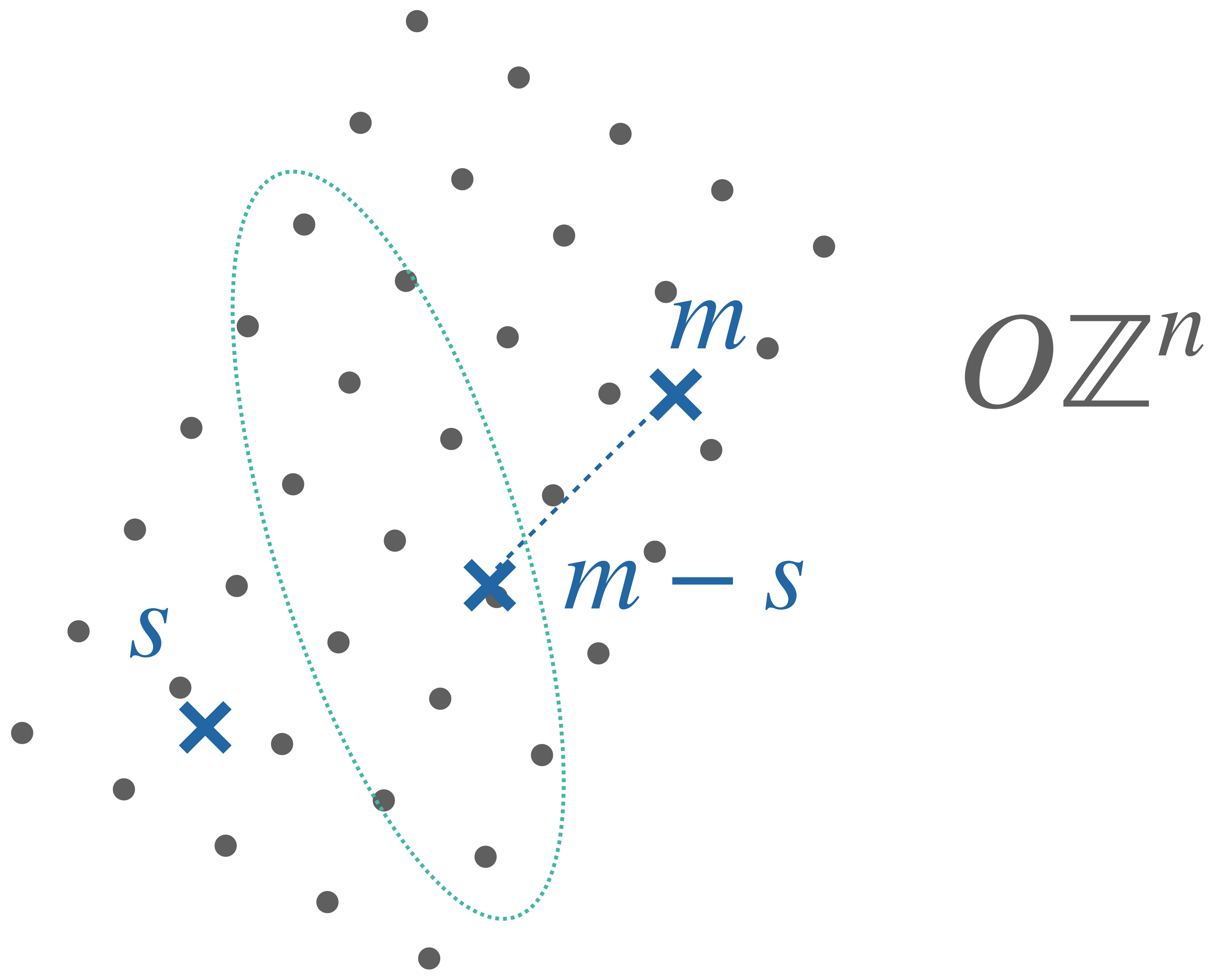

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Prover | **Verifier**

$\tilde{m} = \mathbf{U}m$ ⟵ m

Sample short $\tilde{x}$ in $\mathbf{O}\tilde{m} + \mathbb{Z}^n$

$s = \mathbf{U}^{-1}\mathbf{O}^t\tilde{s} = m - \mathbf{U}^{-1}\mathbf{O}^t\tilde{x}$ ⟶ s

Observation:

Any $\mathbf{V} \in GL_n(\mathbb{Z})$ such that $\mathbf{V}^t\mathbf{V} = \mathbf{G}$ allows to convince the verifier.

(Run the protocol with $\mathbf{V}$ instead of $\mathbf{U}$).

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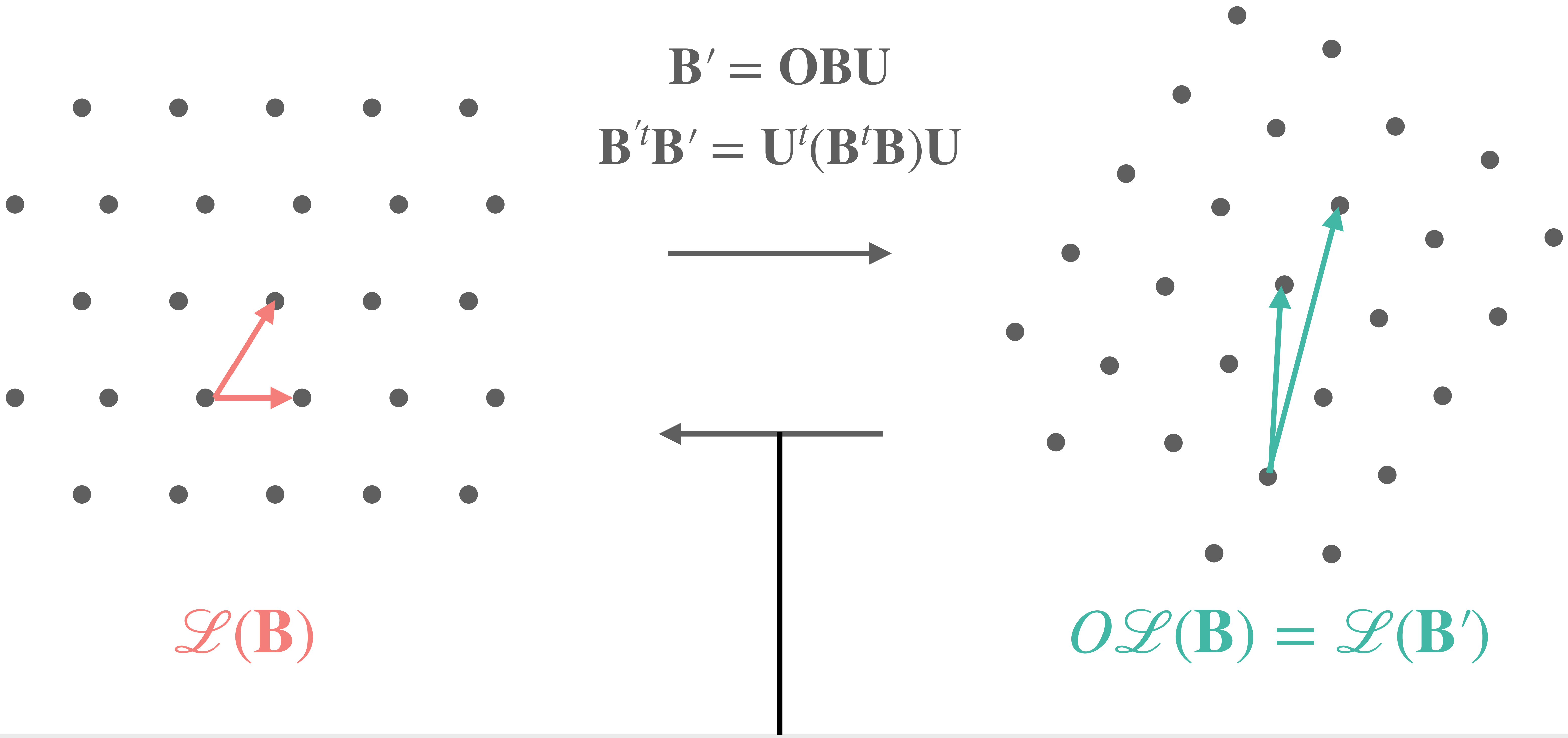
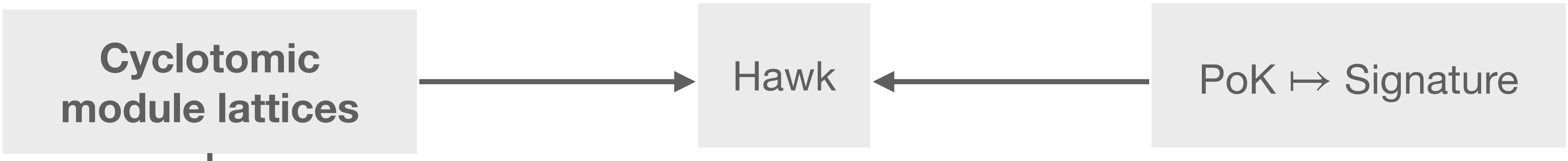
The Lattice Isomorphism Problem (LIP), with quadratic forms



Two quadratic forms $\mathbf{G}, \mathbf{G}'$ are integrally congruent when $\mathbf{G}' = \mathbf{U}^t\mathbf{G}\mathbf{U}$ for some congruence matrix $\mathbf{U} \in \mathrm{GL}_n(\mathbb{Z})$.

LIP$^{\mathbf{B}}$: *Given* $\mathbf{B}$, $\mathbf{G} = \mathbf{B}^t\mathbf{B}$ *and* $\mathbf{G}' \sim_{\mathbb{Z}} \mathbf{G}$, *find any congruence matrix* $\mathbf{U}$ *between* $\mathbf{G}$ *and* $\mathbf{G}'$.

Hawk[1] and module lattices



<u>New context:</u>

$m = 2^\ell$, $K = \mathbb{Q}(\zeta_m)$, and $\mathcal{O}_K := \mathbb{Z}[\zeta_m]$ (for ζ_m primitive).

Identify $\mathbb{Z}^m$ with $\mathcal{O}_K^2$, **a free module lattice of rank 2.**

Transpose becomes conjugate-transpose

Two forms $\mathbf{G}, \mathbf{G}'$ are $\mathcal{O}_K$-congruent when $\mathbf{G}' = \mathbf{U}^*\mathbf{G}\mathbf{U}$ for some congruence matrix $\mathbf{U} \in \mathrm{GL}_n(\mathcal{O}_K)$.

*(Free-)*Mod-LIP$_K^{\mathbf{B}}$: *Given $\mathbf{B}$, $\mathbf{G} = \mathbf{B}^*\mathbf{B}$ and $\mathbf{G}' \sim_{\mathcal{O}_K} \mathbf{G}$, find any congruence matrix $\mathbf{U}$ between $\mathbf{G}$ and $\mathbf{G}'$.*

[1] : https://hawk-sign.info, also ePrint 2022/1155 (L. Ducas, E. Postlethwaite, L. Pulles and W. Van Woerden. See also Wessel's talk!

Hawk[1] and module lattices

Cyclotomic module lattices → **Hawk** ← PoK $\mapsto$ Signature

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Identify $\mathbb{Z}^m$ with $\mathcal{O}_K^2$, **a free module lattice of rank 2.**

Transpose becomes conjugate-transpose

Could be extended to many number fields K

Don't do it in rank 1!

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Why not using ideal lattices?

Say $K = \mathbb{Q}(\zeta)$, a cyclotomic field, the lattice is $\mathcal{O}_K$.

We pick a private unit $u \in \mathcal{O}_K^\times$, and publish the **totally real** element $g = u^* u$.

For all $\sigma : K \to \mathbb{C}$
$\sigma(g) \in \mathbb{R}_+$

Observation:
The congruence class is then the set of solutions of **the relative norm equation**

$$N(x) = g,$$

where $N : K \to F, N(a) = a^* a$.

F is the field fixed by $\cdot^*$
(A totally real field)

Do not use ideal lattice because there are polynomial time algorithms[1] for this!

(But this information is useful for the rest of the talk!)

[1] :Gentry-Szydlo's algorithm for cyclotomic fields, Lenstra-Silverberg for general « CM-orders ».

Now what's the plan for today?

Target: ModLIP **over Rank 2, Free, module** lattices

Mod-LIP$_K^{\mathbf{B}}$: *Given* $\mathbf{B}$, $\mathbf{G} = \mathbf{B}^*\mathbf{B}$ *and* $\mathbf{G}' \sim_{\mathcal{O}_K} \mathbf{G}$, *find any congruence matrix* $\mathbf{U}$ *between* $\mathbf{G}$ *and* $\mathbf{G}'$.

1. Fermat's two squares problem
2. A heuristic polynomial time algorithm to solve ModLIP over totally real number fields
3. Lagrange's four square theorem, quaternions: new reductions for ModLIP over CM-extension fields
4. State of affairs, perspectives, open questions

ModLIP in rank 2
over totally real fields

aka.

Fermat’s « two squares » theorem

The totally real version of ModLip

1

Now, we let K be a totally real number field (all embeddings map to $\mathbb{R}$) and $\mathbf{B} = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$

$$\mathbf{G}_{pub} = \begin{bmatrix} g_0 & \star \\ \star & g_1 \end{bmatrix} = \begin{bmatrix} x^2 + y^2 & \star \\ \star & z^2 + w^2 \end{bmatrix} = \mathbf{B}^* \mathbf{B}$$

So we can recover the key if we can compute all sums of two squares giving g_0, g_1.

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This links back to ***Fermat's two squares theorem:***

- A prime integer p is the sum of two integers squared if and only if $p \equiv 1\ [4]$.
- The set of integers that can be written as the sums of two squares is:

$$S_2(\mathbb{Z}) := \{2^e \cdot \prod_{p \equiv 1\ [4]} p^{v_p} \cdot \prod_{p \equiv 3\ [4]} p^{2v_p} \ : e, v_p \in \mathbb{N}\}$$

We need:

- An **algorithmic** version of it
- An extension to **algebraic integers**

Two classic proofs strategies

$S_2(\mathbb{Z})$ is stable by multiplication. With unique factorization into primes, the main task is understanding primes that are in $S_2(\mathbb{Z})$.

Proof with geometry
We look at $p = x^2 + y^2$, that is $x^2 \equiv -y^2\ [p]$.

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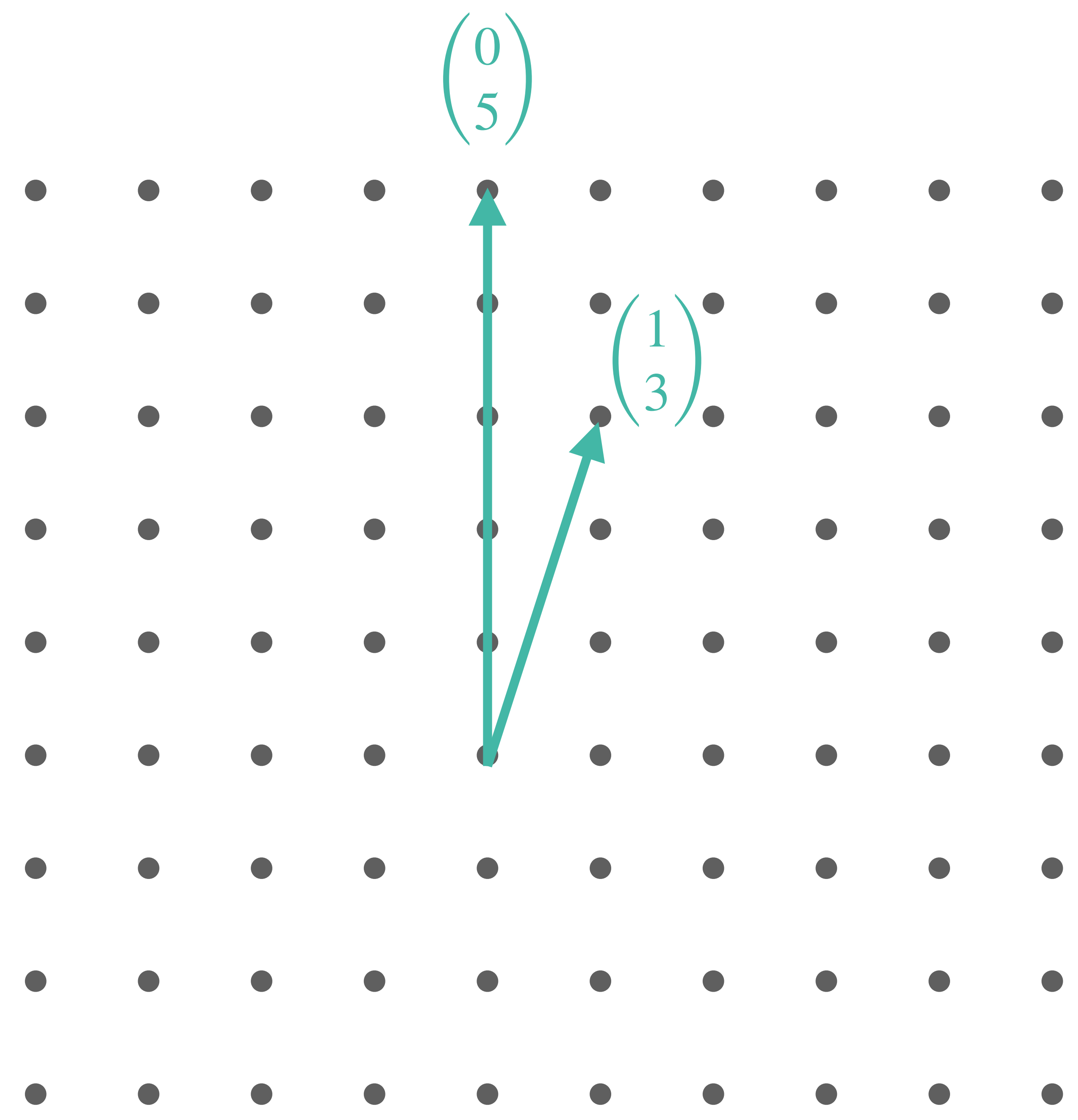
1. -1 is a square mod p $\Leftrightarrow p \equiv 1\,[4]$.
   Let $u^2 \equiv -1\,[p]$.

2. Define $\mathscr{L}(p) = \{(a,b) \in \mathbb{Z}^2 \,:\, au - b \equiv 0\,[p]\}$.
   For $v \in \mathscr{L}(p)$, we have $\|v\|^2 = a^2 + b^2 \in p\mathbb{Z}$.

   A basis is $\begin{bmatrix} 1 & 0 \\ u & p \end{bmatrix}$.



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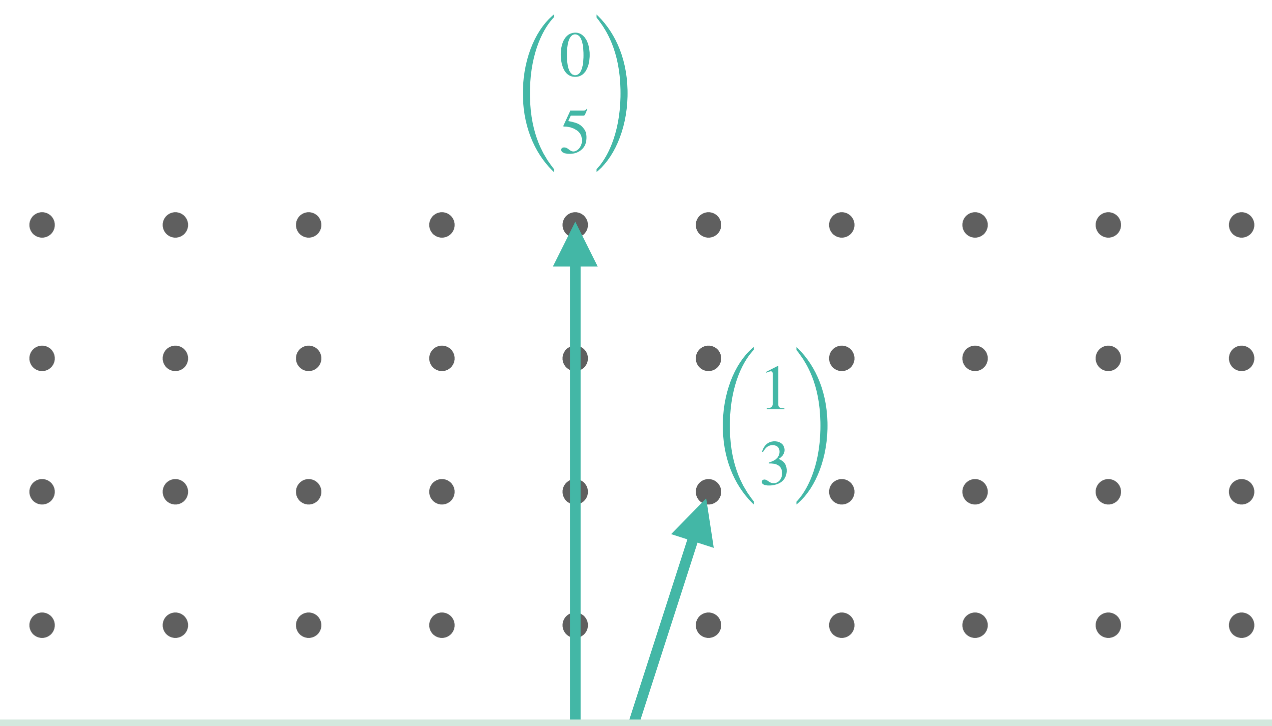

Minkowski's theorem (in rank 2)
Let $\mathscr{L}$ be a lattice of rank 2. The shortest vector in $\mathscr{L}\setminus\{0\}$ has length:

$$\lambda_1(\mathscr{L})^2 \leq \frac{4}{\pi} \cdot \det(\mathscr{L}).$$

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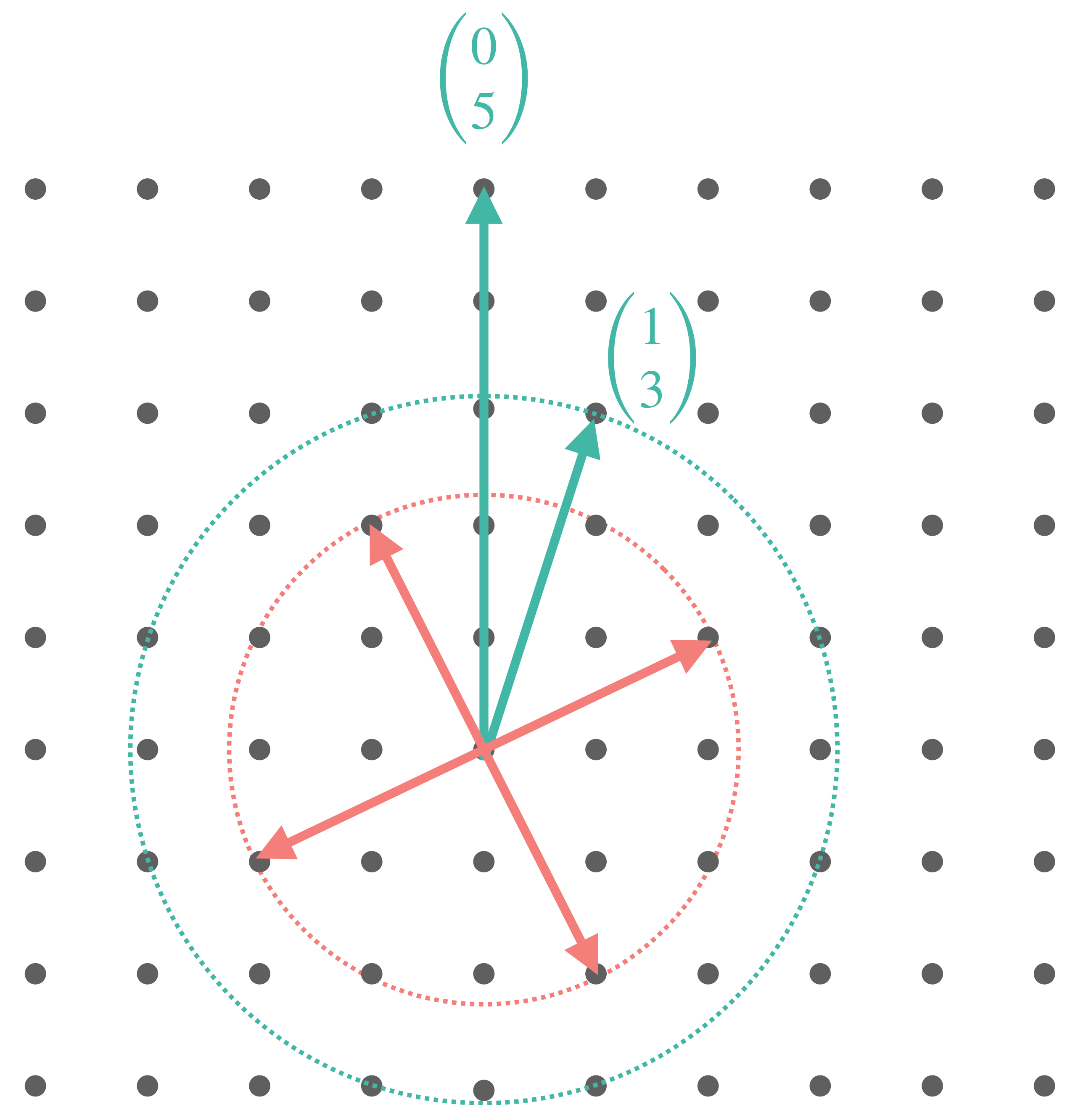
3. Any shortest vector gives a two-square sum for p.
   Compute them with Gauss-Lagrange's algorithm.



($\mathscr{L}(p)$ is similar to $\mathbb{Z}^2$)

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Proof with arithmetic

In the Gauss integers $\mathbb{Z}[i]$, we have:

$$p = N(x+iy) := (x+iy)(x-iy).$$

1. p factors $\Leftrightarrow T^2 + 1$ factors modulo p
   Its discriminant is $\Delta = -4$. It is a square mod p iff -1 is a square modulo p.

Two classic proofs strategies

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   Two conjugate primes above p. One is $\mathfrak{p} = \langle p, i-a \rangle$.

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3. $\mathbb{Z}[i]$ is *Euclidean*: compute gcd of p and $i-a$ with Euclidean division to obtain a generator $x+iy$ of $\mathfrak{p}$.
4. Do the same for $\mathfrak{p}^*$, loop over all units in $\mathbb{Z}[i]$ to get all generators.

Two classic proofs strategies

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Proof with geometry	Proof with arithmetic
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3. Any shortest vector gives a two-square sum for p. Compute them with Gauss-Lagrange's algorithm.	3. Compute generators of $\mathfrak{p}, \mathfrak{p}^*$. Products of them and units give two-square sums.

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3. Any shortest vector gives a two-square sum for p.
   Compute them with Gauss-Lagrange's algorithm.

Gauss-Lagrange is very similar to Euclidean division

Reciprocity

Proof with arithmetic

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2. Compute $p\mathbb{Z}[i] = \mathfrak{p}\mathfrak{p}^*$ by factoring $T^2 + 1 \bmod p$.
3. Compute generators of $\mathfrak{p}, \mathfrak{p}^*$.
   Products of them and units give two-square sums.

This works because Euclidean ⇒ ***Principal*** ⇒UFD.

Extension to totally real fields

Let $F = \mathbb{Q}(\zeta + \zeta^{-1})$, with ζ a primitive root of unity, and $\mathcal{O}_F = \mathbb{Z}[\zeta + \zeta^{-1}]$.

We have $\alpha = x^2 + y^2$, for some $x, y \in \mathcal{O}_F$. ***No unique factorization anymore.***

Instead we have unique factorization in **prime ideals**: $\alpha\mathcal{O}_F = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}}$.

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Analog of $\mathbb{Q}(i)$ is $F(i)$, and reciprocity is now factoring $T^2 + 1$ modulo $\mathfrak{p}$.

That is, $\mathfrak{p}$ splits when Δ is a square in the finite field $\mathcal{O}_F/\mathfrak{p}$.

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Above, we have $\alpha\mathcal{O}_{F(i)} = (x + iy)\mathcal{O}_{F(i)} \cdot (x - iy)\mathcal{O}_{F(i)}$.

$$\mathfrak{p}\mathcal{O}_{F(i)} = \begin{cases} \mathfrak{P}\mathfrak{P}^* & \text{, if } \mathfrak{p} \text{ splits} \\ \mathfrak{p} & \text{, if } \mathfrak{p} \text{ is inert} \\ (\mathfrak{P}^2 & \text{, if } \mathfrak{p} \text{ ramifies)} \end{cases}$$

These ideals: 1) must share the prime factors of α
2) have **conjugated** prime factors.

This implies $\alpha\mathcal{O}_{F(i)} = \prod_{\mathfrak{p}\ splits} (\mathfrak{P}\mathfrak{P}^*)^{v_\mathfrak{P}} \cdot \prod_{\mathfrak{p}\ inert} \mathfrak{p}^{2v_\mathfrak{p}}$.

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Above, we have $\alpha\mathcal{O}_{F(i)} = (x+iy)\mathcal{O}_{F(i)} \cdot (x-iy)\mathcal{O}_{F(i)}$

$$= \prod_{\mathfrak{p}\ splits} (\mathfrak{P}\mathfrak{P}^*)^{v_\mathfrak{P}} \cdot \prod_{\mathfrak{p}\ inert} \mathfrak{p}^{2v_\mathfrak{p}}.$$

$$\mathfrak{p}\mathcal{O}_{F(i)} = \begin{cases} \mathfrak{P}\mathfrak{P}^* & \text{, if } \mathfrak{p} \text{ splits} \\ \mathfrak{p} & \text{, if } \mathfrak{p} \text{ is inert} \\ (\mathfrak{P}^2 & \text{, if } \mathfrak{p} \text{ ramifies)} \end{cases}$$

Theorem (up to ramification):
The set of elements in $\mathcal{O}_F$ that can be written as the sum of two $\mathcal{O}_F$-squares is

$$S_2(\mathcal{O}_F) = \{\alpha \in \mathcal{O}_F \,:\, \alpha\mathcal{O}_F = \prod_{\mathfrak{p}\ splits} \mathfrak{p}^{v_\mathfrak{p}} \cdot \prod_{\mathfrak{p}\ inert} \mathfrak{p}^{2v_\mathfrak{p}}\}$$

From two-squares to module lattices isomorphisms

Computing sums of squares

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Observations:

1) We can compute these primes given α
2) Must be at least **one principal ideal** $(x + iy)\mathcal{O}_{F(i)}$ among all meaningful products of these primes

Computing sums of squares

2

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Observations:

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To test if an ideal is principal in number fields and to compute a generator is a (classically) hard problem!

And we may not even find the correct generator…

Computing sums of squares

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Observations:

1) We can compute these primes given α
2) Must be at least **one principal ideal** $(x+iy)\mathcal{O}_{F(i)}$ among all meaningful products of these primes
3) We know the relative norm $N_{F(i)|F}(x+iy) = \alpha$.

Recover generators up to **roots of unity**

Computing sums of squares

2

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Recover generators up to **roots of unity**

Gentry-Szydlo's algorithm:

There is a **polynomial time** algorithm that, given a basis of an ideal I in a cyclotomic field, and a candidate β for the relative norm of a potential generator g of I:

1) Decides if I is principal;
2) If it is, returns an element $g' = \rho g$ where ρ is a root of unity in the field.

Computing sums of squares

2

$$S_2(\mathcal{O}_F) = \{\alpha \in \mathcal{O}_F \,:\, \alpha\mathcal{O}_F = \prod_{\mathfrak{p}\ splits} \mathfrak{p}^{v_\mathfrak{p}} \cdot \prod_{\mathfrak{p}\ inert} \mathfrak{p}^{2v_\mathfrak{p}}\}$$

Observations:

1) We can compute these primes given α
2) Must be at least **one principal ideal** $(x+iy)\mathcal{O}_{F(i)}$ among all meaningful products of these primes
3) We know the relative norm $N_{F(i)|F}(x+iy) = \alpha$.
4) We can also compute the roots of unity in $F(i)$.

Recover all **useful** generators, in **polynomial time,** by solving **relative norm equations**

An algorithm to compute sums-of-squares

Input: $\alpha \in \mathcal{O}_F$

Output : the set $S_2(\alpha)$ of all possible $(x, y) \in \mathcal{O}_F^2$ such that $x^2 + y^2 = \alpha$.

1. Factor $\alpha\mathcal{O}_F = \prod_{splits} \mathfrak{p}^{v_\mathfrak{p}} \cdot \prod_{inerts} \mathfrak{q}^{v_\mathfrak{q}}$; set $\mathcal{S} = \emptyset$; if one $v_\mathfrak{q}$ is not even, return $\mathcal{S}$.
2. For all $0 \leq e_\mathfrak{p} \leq v_\mathfrak{p}$, do:
   a. Compute $I = \prod_{splitters} \mathfrak{P}^{e_\mathfrak{P}}(\mathfrak{P}*)^{v_\mathfrak{P}-e_\mathfrak{P}} \cdot \prod_{inerts} \mathfrak{q}^{v_\mathfrak{q}/2}$ and set $\mathcal{G} = \emptyset$
   b. $g \leftarrow$ GentrySzydlo(I, α);
   c. If $g \neq \perp$, set $\mathcal{G} = \{\rho \cdot g \ : \ \rho$ root of unity in $F(i)\}$.
   d. For all $g' \in \mathcal{G}$, write $g' = x + iy$ and $\mathcal{S} = \mathcal{S} \cup \{(x, y)\}$
3. Return $\mathcal{S} \cap \mathcal{O}_F^2$.

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3. Return $\mathcal{S} \cap \mathcal{O}_F^2$.

Not polynomial time if the factorization is not given.

Possibly **many** combinations

$\mathcal{O}_F + i\mathcal{O}_F \subsetneq \mathcal{O}_{F(i)}$ in general

An algorithm to solve totally real modLIP

2

Input: the public Gram matrix $\mathbf{G} = \mathbf{B}^t\mathbf{B} = \begin{bmatrix} g_0 & \star \\ \star & g_1 \end{bmatrix}$ and a matrix $\mathbf{C}$ such that $\mathbf{C}^t\mathbf{C} = \mathbf{U}^t\mathbf{G}\mathbf{U}$

Output : the set of $\mathbf{U} \in GL_2(\mathcal{O}_F)$ describing the congruence class of $\mathbf{G}$

1. For $b \in \{0,1\}$:
   a. $\mathcal{S}_b \leftarrow$ TwoSquares(g_b)
2. Let $\mathcal{U} = \emptyset$.
   For all $(a, b), (a', b') \in \mathcal{S}_0 \times \mathcal{S}_1$:
   a. $\mathbf{D} \leftarrow \begin{bmatrix} a & a' \\ b & b' \end{bmatrix}$
   b. If $\mathbf{V} = \mathbf{C}^{-1}\mathbf{D}$ is a congruence matrix for $\mathbf{G}$, set $\mathcal{U} = \mathcal{U} \cup \{\mathbf{V}\}$.
3. Return $\mathcal{U}$.

Theorem (Mureau, Pellet—Mary, Pliatsok, W.)

This algorithm returns (a description of) the congruence class of $\mathbf{G}$.

Possibly **many** steps.

Towards polynomial time: the randomization step (1/2)

Goal: avoid factoring and control loops to achieve (classic) polynomial time

With $\mathbf{G} = \mathbf{B}^t\mathbf{B}$, from vectors in $\mathcal{O}_F^2$ we can learn the norm of vectors in $\mathcal{L}(\mathbf{B})$:

$$(x, y)\mathbf{G}(x, y)^t = (x, y)\mathbf{B}^t \cdot \mathbf{B}(x, y)^t = a^2 + b^2$$

If we have two that are linearly independent, we deduce congruence matrices by linear algebra:

$$\mathbf{D} = \mathbf{CV} \sim \mathbf{D}' = \mathbf{CVX}$$

$$(a, b) = \mathbf{B}(x, y)$$

$$\mathbf{X} = \begin{bmatrix} x & x' \\ y & y' \end{bmatrix}$$

<u>Lemma:</u> we can sample Gaussians $(x, y) \in \mathcal{O}_F^2$ so that $\mathbf{B}(x, y)$ is spherical, without knowing $\mathbf{B}$.

(This way we have at least *some* control over (a, b))

Towards polynomial time: the randomization step (2/2)

Goal: avoid factoring and control loops to achieve (classic) polynomial time

Randomization step: feed *random* vectors $(x, y) \in \mathcal{O}_F^2$ to $\mathbf{G}$ until:

- Two of them span the space
- These two have a prime relative norm, that is, $(x, y)\mathbf{G}(x, y)^t = a^2 + b^2$ is a prime in $\mathcal{O}_F$.

$\Rightarrow$ compute the corresponding sum of squares **without having to factor!**

$\Rightarrow$ primes in F have at most two divisors in $F(i)$ so poly($[F : \mathbb{Q}]$) steps in the loop at worst.

Heuristic assumption:

With large enough width, $q := a^2 + b^2$ behaves like a « uniformly random » principal ideal.

(GRH) Proba$\big(q$ is prime$\big) \approx \dfrac{1}{\rho_F \cdot \ln N(q)}$, ρ_F residue at 1 of the Dedekind zeta function of F.

Conclusion for modLIP over totally real fields

Theorem (Mureau, Pellet—Mary, Pliatsok, W.)
Let F be a totally real number field with ring of integers $\mathcal{O}_F$.
There is an algorithm that solves modLIP over rank 2 free $\mathcal{O}_F$-modules in heuristic polynomial-time (in ρ_F, $[F:\mathbb{Q}]$).

- The full algorithm is implemented for cyclotomic fields with conductor $m = 4k$.
  https://gitlab.inria. fr/capsule/code-for-module-lip

- In the paper[1], we provide an algorithm for rank 2 (non-free) modules and its tools. It also runs in polynomial-time (depending on an additional, precomputable quantity).

[1] : ePrint 2024/441

ModLIP in rank 2, over CM-extensions

Alexandre Wallet, Maths for PQC workshop, 5/08/2024

Back to the general case (or almost)

For simplicity: let $K = \mathbb{Q}(\zeta)$ where ζ is a primitive root of unity, and $F = \mathbb{Q}(\zeta + \zeta^{-1})$. Assume $i \in K$.

$$\mathbf{G}_{pub} = \begin{bmatrix} g_0 & \star \\ \star & g_1 \end{bmatrix} = \begin{bmatrix} x^*x + y^*y & \star \\ \star & z^*z + w^*w \end{bmatrix} = \mathbf{B}^*\mathbf{B}$$

We can write $x = x_{\mathbb{R}} + ix_{\mathbb{I}} \in K = F + iF$ and $x^*x = x_{\mathbb{R}}^2 + x_{\mathbb{I}}^2$ (and similarly for y, z, w).

So we can recover $\mathbf{B}$ if we can compute all **sums of four squares** that give g_0, g_1.

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So we can recover $\mathbf{B}$ if we can compute all **sums of four squares** that give g_0, g_1.

This links to **Lagrange's four square theorem:** Every integer can be written as the sum of four integers squared.

At least two proofs:

- a **geometric** one with short vectors (mostly for prime integers)
- an algebraic proof using **quaternions**

We need:

- An **algorithmic** version of it
- An extension to **cyclotomic integers**

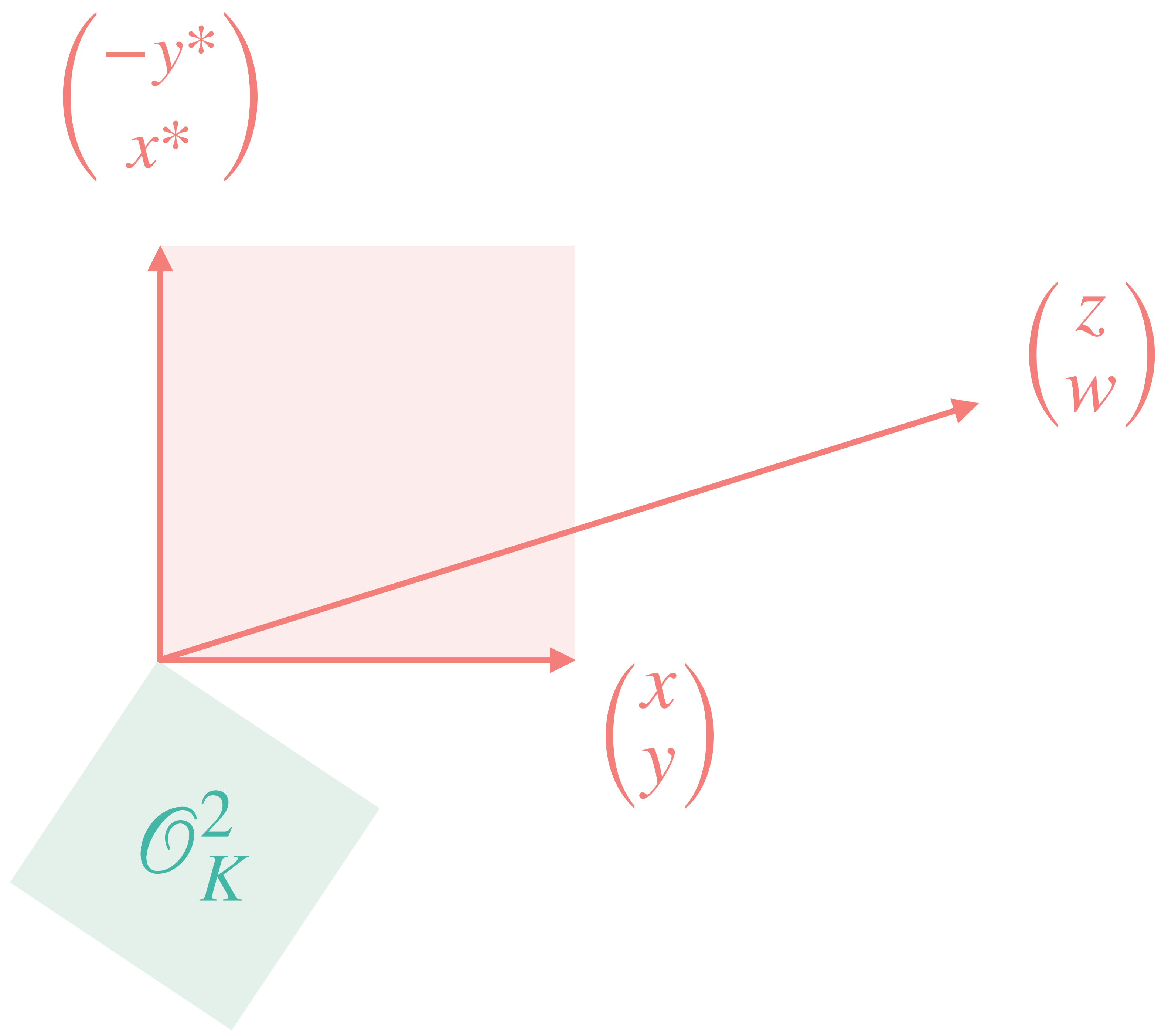
The nice case: cyclotomic modLIP over $\mathcal{O}_K^2$ (1/2)

Geometric view[1]

$\mathbf{G} = \mathbf{B}^*\mathbf{B}$ with $\mathbf{B} = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ and $\det \mathbf{B} = 1$ (a basis of $\mathcal{O}_K^2$)

Another interesting basis: $\mathbf{S} = \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix}$.

It is essentially unitary: $\mathbf{S}^*\mathbf{S} = g_0 \cdot \mathbf{I}_2$



[1] : Thomas and Heorhii, ePrint 2024/1148

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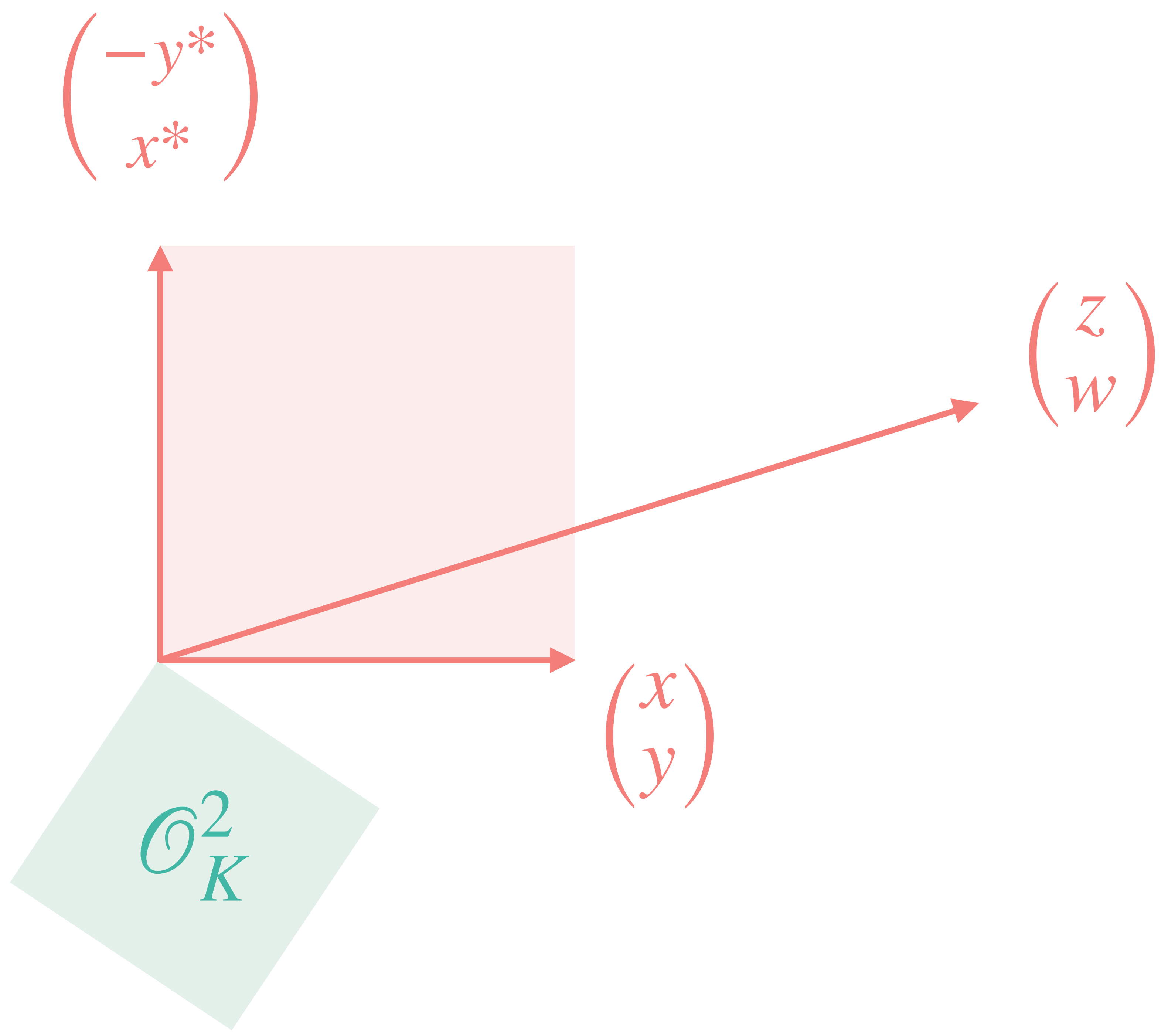
It is essentially unitary: $\mathbf{S}^*\mathbf{S} = g_0 \cdot \mathbf{I}_2$.

Coordinate-wise: $\mathbf{B}^{-1}\mathbf{S} = \begin{bmatrix} 1 & -g_{01} \\ 0 & g_0 \end{bmatrix} =: \mathbf{T}$

$\Rightarrow \mathscr{L}(\mathbf{T}^*)$ is a public hypercubic lattice, and it has a secret orthogonal basis $\mathbf{S}^*$.



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The nice case: cyclotomic modLIP over $\mathcal{O}_K^2$

Geometric view[1]

The lattice spanned by $\mathbf{T}^* = \begin{bmatrix} 1 & 0 \\ -g_{01}^* & g_0 \end{bmatrix}$ is similar to lattices from two-squares:

$$\mathcal{L}(g_0) = \{(a, b) \in \mathcal{O}_K^2 \ : \ g_{01}^* a - b = 0 \ [g_0]\}$$

where $g_{01}^* g_{01} \equiv -1 \ [g_0]$: -1 is a two-squares sums mod g_0

Conclusion[1]:

An oracle to compute an orthogonal basis in a hyper cubic lattice recovers (x, y).

Observation:

$\mathbf{S} = \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix}$ is actually a matrix representation for a **quaternion.**

[1] : Thomas and Heorhii, ePrint 2024/1148

Sprinkles of quaternion algebra

Let $i^2 = -1$, and j such that $-ji = ij =: k$ and $j^2 = k^2 = -1$.

Quaternion algebra	$\mathscr{A} := F\langle i, j\rangle$	$\sim F + iF + jF + kF$
	$\cup$	
CM-extension of F	$K = F(i)$	$\sim F + iF$
	$\cup$	
Totally real number field	F	

There is an involution extending the conjugation:

$$(a + ib + jc + kd)^* = a - ib - jc - kd$$

There is a norm map extending the relative norm from $F(i) \mid F$:

$$\mathrm{Nrd}(a + ib + jc + kd) = a^2 + b^2 + c^2 + d^2 \in F$$

Using that $\mathscr{A} = K + Kj$, we can represent elements as matrices of multiplication:

$$x + yj \mapsto \begin{bmatrix} x & -y^* \\ y & x^* \end{bmatrix}$$

Now that we have more algebra:

$$\mathbf{G}_{pub} = \begin{bmatrix} g_0 & g_{01} \\ \cdot & g_1 \end{bmatrix} = \begin{bmatrix} x^*x + y^*y & x^*z + y^*w \\ \cdot & z^*z + w^*w \end{bmatrix} = \mathbf{B}^*\mathbf{B}$$

Observations:

- We have not used the anti-diagonal term g_{01} yet
- As we reduced to sums-of-squares computations, maybe we can mimic the previous strategy

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The non commutative-settings brings a lot of inconveniences:

	$F(i)$	$F\langle i, j\rangle$
Unique factorisation in prime ideals	Yes	Not for sided ideals
Maximal ring of integers	1	Many
Roots of unity	Straightforward	Depends on the subring
Algorithms for norm equations	Some poly-time	All known exp-time

The nice case: cyclotomic modLIP over $\mathcal{O}_K^2$ (2/2)

Geometric view[1]

The lattice spanned by $\mathbf{T}^* = \begin{bmatrix} 1 & 0 \\ -g_{01}^* & g_0 \end{bmatrix}$ is similar to lattices from two-squares:

$$\mathcal{L}(g_0) = \{(a,b) \in \mathcal{O}_K^2 \ : \ g_{01}^* a - b = 0 \ [g_0]\}$$

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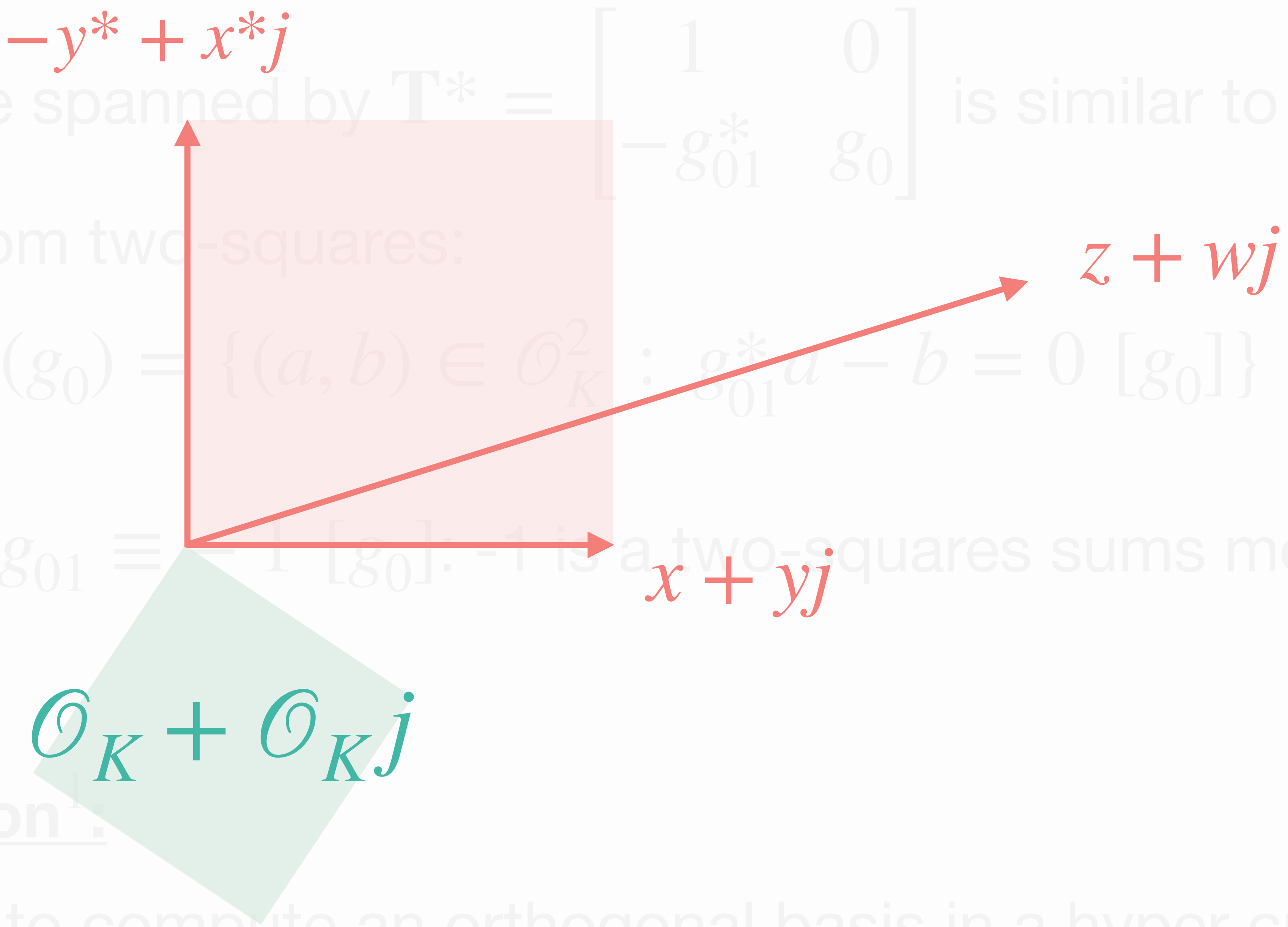

Conclusion[1]:

An oracle to compute an orthogonal basis in a hyper cubic lattice recovers (x, y).

Quaternion view[2]

- $\mathcal{O}_K^2$ identifies to the maximal order $\mathcal{O}_K + \mathcal{O}_K j$.
  It is generated by $\alpha = x + yj$ and $\beta = z + wj$.
- The matrix $\mathbf{S}$ represents the quaternion $x + yj$, so:

$$\mathbf{S}^{-1}\mathbf{B} = \mathbf{T} \ \Rightarrow \alpha^{-1}\beta = g_0^{-1}(g_{01} + j)$$

Lemma: we can compute a basis of $\alpha(\mathcal{O}_K + \mathcal{O}_K j)$ from the public data.

[1] : Thomas and Heorhii, ePrint 2024/1148

[2] : with Clémence, Guilhem, Alice and Pierre-Alain, ePrint 2024/1147

The nice case: cyclotomic modLIP over $\mathcal{O}_K^2$ (2/2)

Geometric view[1]

The lattice spanned by $\mathbf{T}^* = \begin{bmatrix} 1 & 0 \\ -g_{01}^* & g_0 \end{bmatrix}$ is similar to lattices from two-squares:

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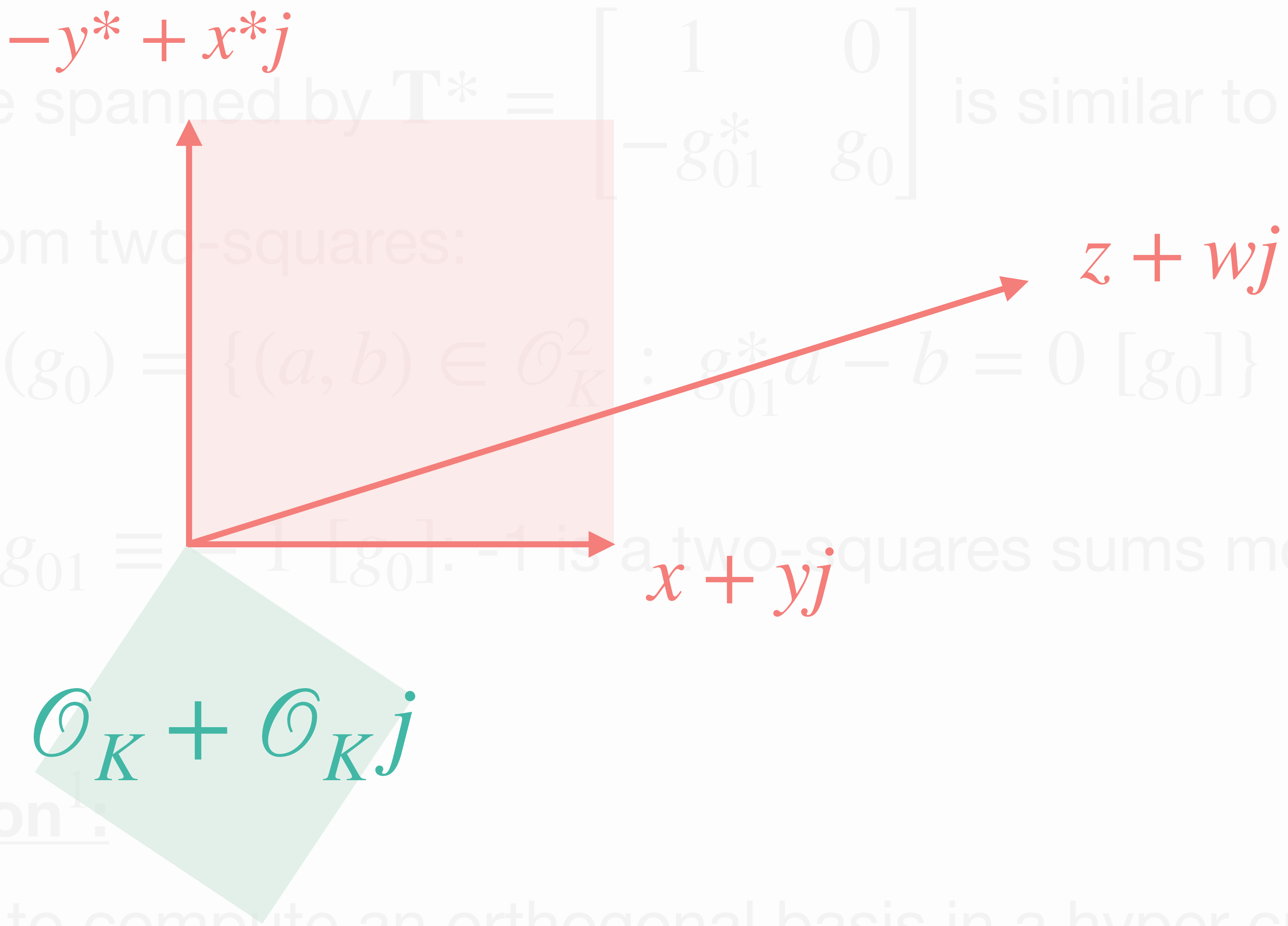

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<u>Lemma:</u> we can compute a basis of $\alpha(\mathcal{O}_K + \mathcal{O}_K j)$ from the public data.

We have the norm of α since $\mathrm{Nrd}(\alpha) = g_0$.

$\Rightarrow$ **we recover α if we can compute generators of principal ideals given their relative norm.**

[1] : Thomas and Heorhii, ePrint 2024/1148

[2] : with Clémence, Guilhem, Alice and Pierre-Alain, ePrint 2024/1147

The nice case: cyclotomic modLIP over $\mathcal{O}_K^2$

3

Geometric view[1]	Quaternion view[2]

Geometric view[1]

The lattice spanned by $\mathbf{T}^* = \begin{bmatrix} 1 & 0 \\ -g_{01}^* & g_0 \end{bmatrix}$ is hyper-cubic:

$$\mathscr{L}(g_0) = \{(a,b) \in \mathcal{O}_K^2 \ : \ g_{01}^* a - b = 0 \ [g_0]\}$$

Conclusion[1]:

An oracle to compute an orthogonal basis in a hyper cubic lattice recovers (x, y).

*(We do not know efficient algorithms to compute orthogonal bases **in large dimension**.)*

Quaternion view[2]

The public data gives a principal ideal $(x + yj)\mathcal{O}$, and we know the reduced norm of this generator.

Conclusion[2]:

Cyclotomic modLIP over $\mathcal{O}_K^2$ reduces to the quaternion version of the Principal Ideal with Relative Norm Problem.

*(We do not know an extension of Gentry-Szydlo's algorithm for this **non-commutative setting**)*

[1] : Thomas and Heorhii, ePrint 2024/1148

[2] : with Clémence, Guilhem, Alice and Pierre-Alain, ePrint 2024/1147

Conclusion for modLIP over CM-extensions

Theorem(s) ([insert the list of peeps]):

Let $K \mid F$ be a CM-extension with ring of integers $\mathcal{O}_K$.

ModLIP over rank 2 free $\mathcal{O}_K$-modules reduces to:
- Computing short orthogonal bases of hypercubic lattices
- Solving the Principal (Left)-ideal problem given the reduced norm of a generator (in quaternion algebras)

- In ePrint 2024/1147, we extend the reduction to rank 2 non-free modules.

- The reduction technique works as well directly over F totally real:
  - ✓No need for randomization anymore
  - ✓This gives a *provable* polynomial time algorithm for totally real modLIP!
    (see also an independent work[1] for an equivalent result with a different approach)

[1] : H. Luo, K. Jiang, Y. Pan and A. Wang, ePrint 2024/1173

Time to wrap-up!

Another reduction (ePrint 2024/1173)

Theorem[1]**:** Let $K \mid F$ be a CM-extension. If an additional symplectic automorphism ϕ_j of $\mathcal{O}_K^2$ is given, then free-modLIP$_K^{\mathbf{B}}$ can be solved in polynomial time.

Very high level idea:

- Knowing ϕ_j , one can construct a CM-order $\mathfrak{O}$ in which $\mathcal{O}_K^2$ is a principal ideal lattice.
- **Lenstra-Silverberg's**[2] applies: it computes a generator of $\mathcal{O}_K^2$ in polynomial time.
- This generator essentially corresponds to a column of the secret basis, up to an isometry of $\mathcal{O}_K^2$.
- (Consequence of Kronecker) Isom($\mathcal{O}_K^2$) is a known finite group of polynomial size.

[1] : H. Luo, K. Jiang, Y. Pan and A. Wang, ePrint 2024/1173

[2] : H. Lenstra, A. Silverberg: arXiv 1706.07373

Another reduction (ePrint 2024/1173)

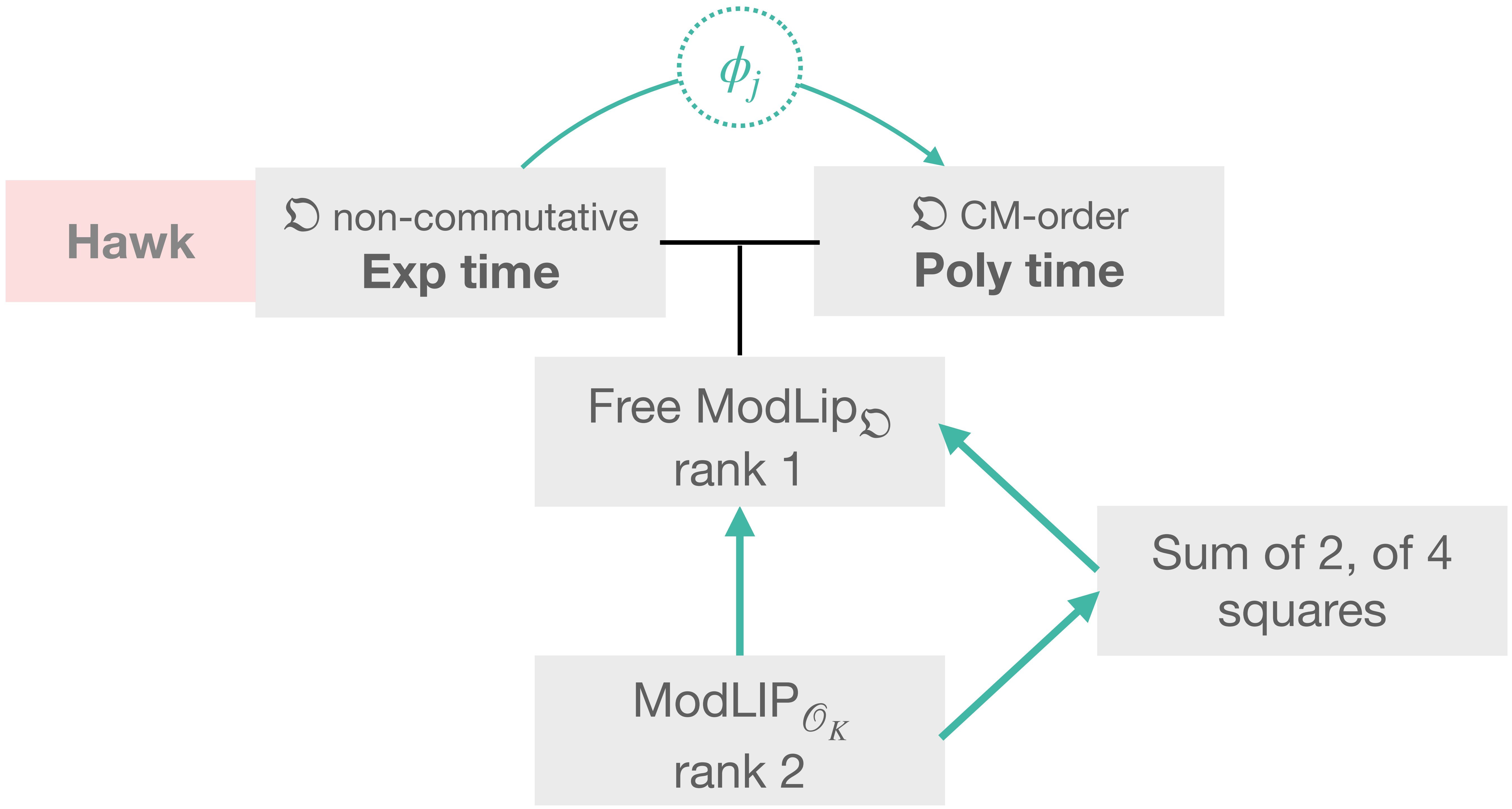
Theorem[1]**:** Let $K | F$ be a CM-extension. If an additional symplectic automorphism ϕ_j of $\mathcal{O}_K^2$ is given, then free-modLIP$_K^{\mathbf{B}}$ can be solved in polynomial time.

My very rough understanding is:

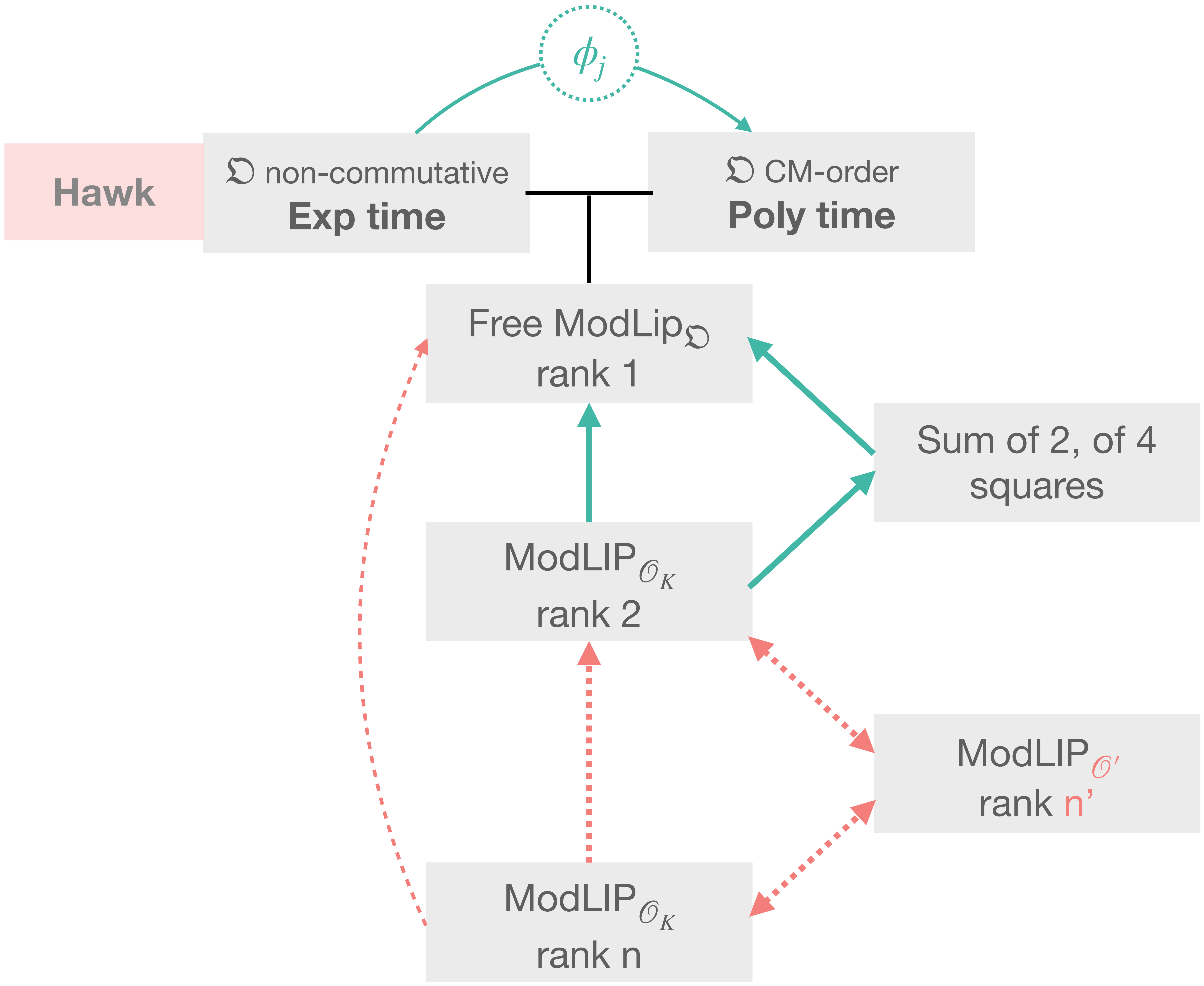
- ϕ_j corresponds to j in the quaternion.
- We need to know its action over the secret basis, but we only know its action in « the canonical one ».
- The article shows that one can easily compute the action of $\tau := [j] \circ \cdot^*$ in the secret basis from public data.

[1] : H. Luo, K. Jiang, Y. Pan and A. Wang, ePrint 2024/1173

[2] : H. Lenstra, A. Silverberg: arXiv 1706.07373


ϕ_j
Hawk
$\mathfrak{O}$ non-commutative
Exp time
$\mathfrak{O}$ CM-order
Poly time
Free ModLip$_{\mathfrak{O}}$
rank 1
Sum of 2, of 4
squares
ModLIP$_{\mathcal{O}_K}$
rank 2

4



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Sum of 2, of 4
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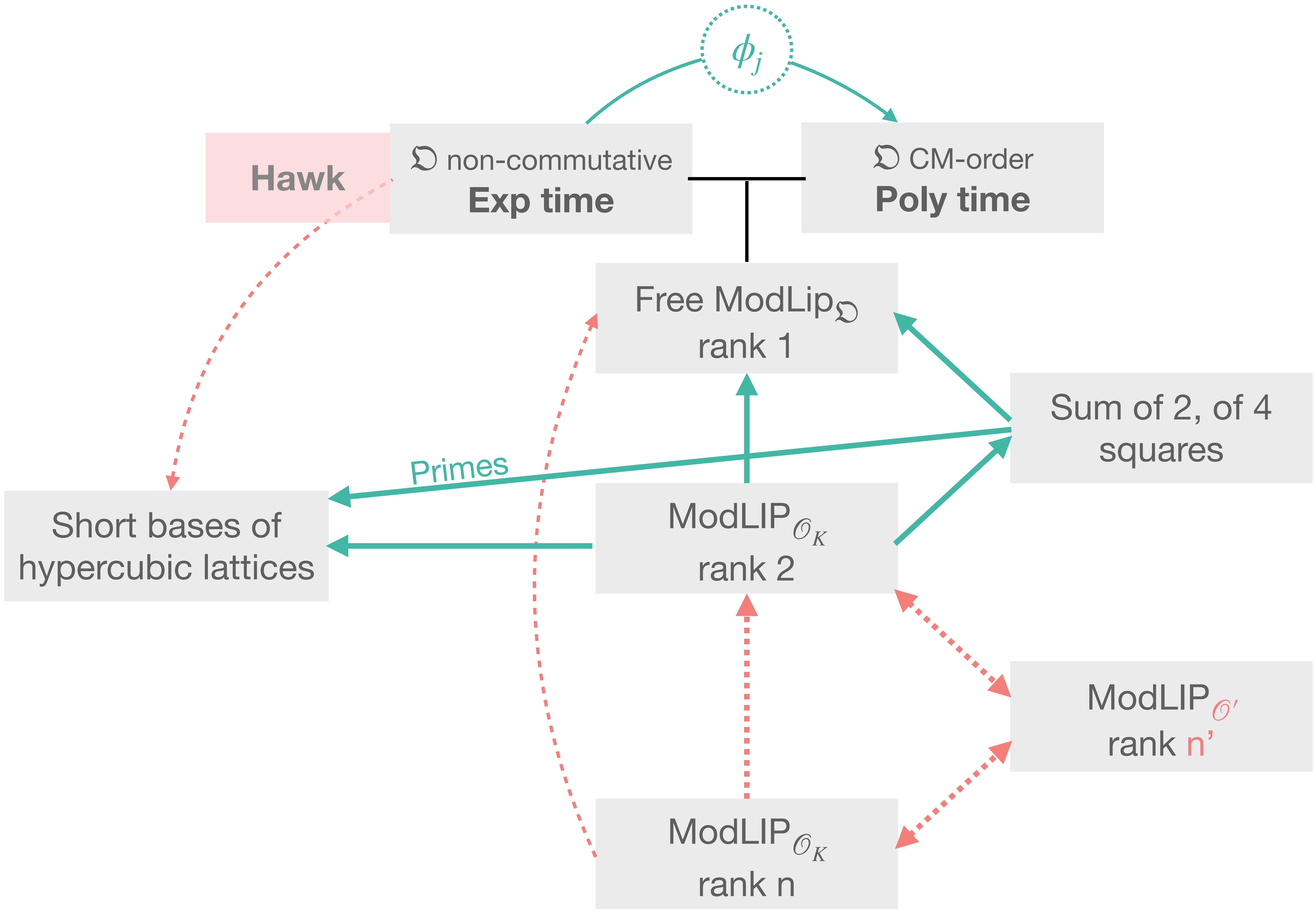
rank 2
ModLIP$_{\mathcal{O}'}$
rank n'
ModLIP$_{\mathcal{O}_K}$
rank n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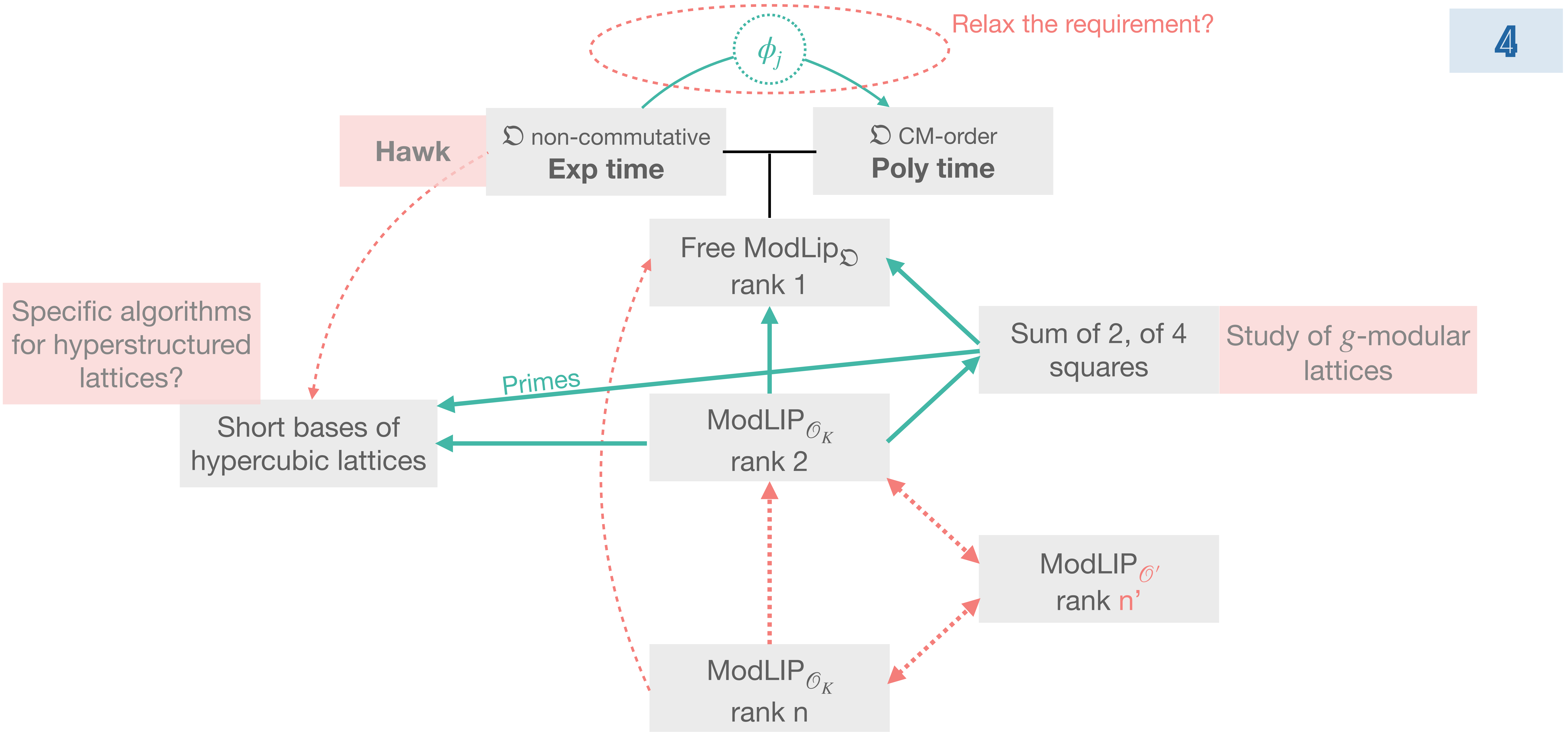
ModLIP$_{\mathcal{O}_K}$
rank n



Relax the requirement?
ϕ_j
4
Hawk
non-commutative
Exp time
CM-order
Poly time
Free ModLip
rank 1
Specific algorithms for hyperstructured lattices?
Sum of 2, of 4 squares
Study of g-modular lattices
Primes
Short bases of hypercubic lattices
ModLIP$_{\mathcal{O}_K}$
rank 2
ModLIP$_{\mathcal{O}'}$
rank n'
ModLIP$_{\mathcal{O}_K}$
rank n

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ModLIP$_{\mathcal{O}_K}$
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ModLIP$_{\mathcal{O}}$
rank n'

ModLIP$_{\mathcal{O}_K}$
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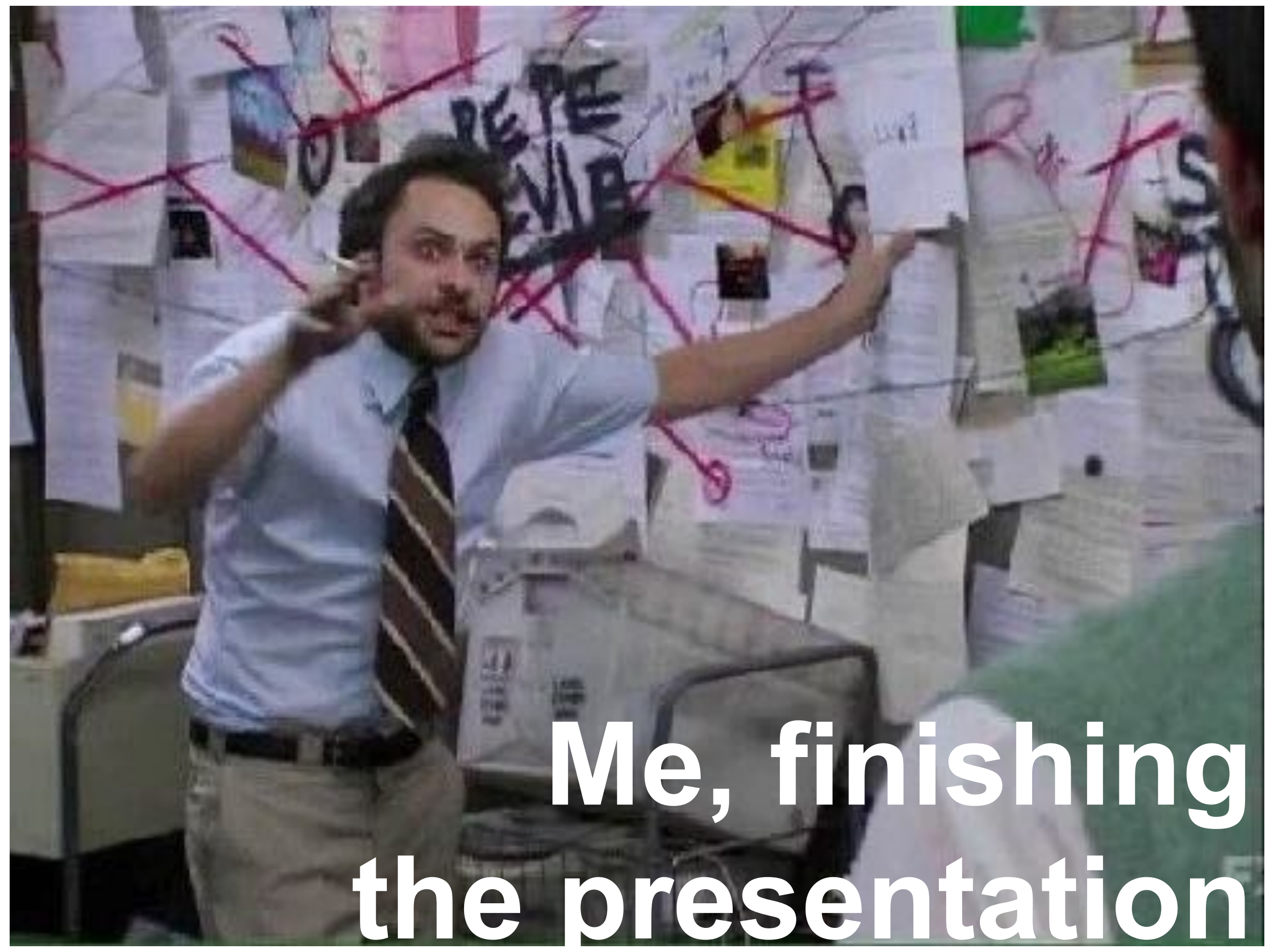

THANK YOU!