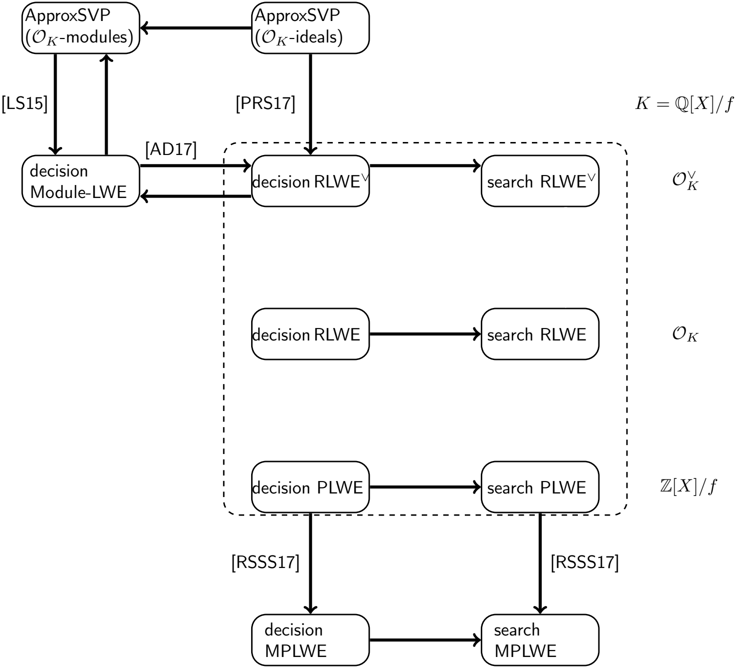
On the Ring-LWE and Polynomial-LWE problems

Miruna Roșca, Damien Stehlé, **Alexandre Wallet**


ApproxSVP ($\mathcal{O}_K$-modules)
ApproxSVP ($\mathcal{O}_K$-ideals)
[LS15]
[PRS17]
$K = \mathbb{Q}[X]/f$
[AD17]
decision Module-LWE
decision RLWE$^\vee$
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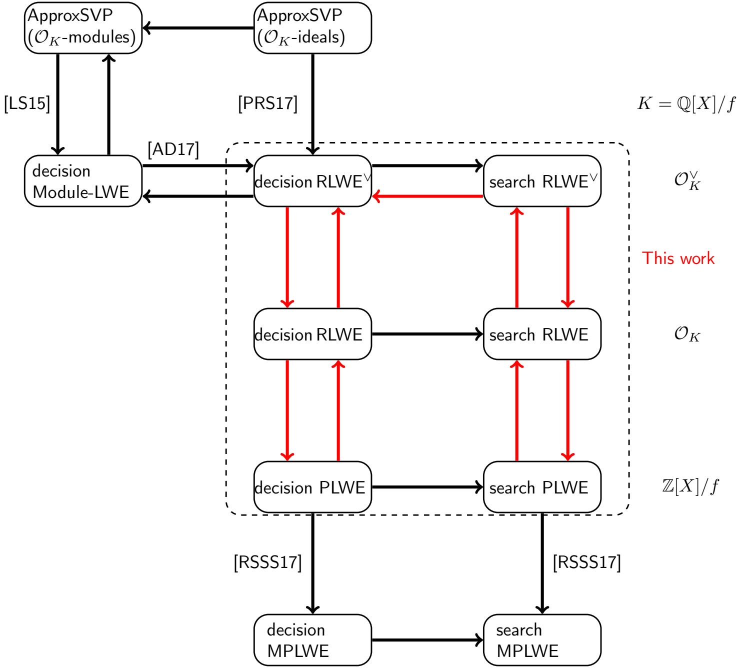

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“On variants of Polynomial-LWE and Ring-LWE”
(EUROCRYPT 2018)

Results:
(A) The 3 settings are essentially† the same
(B) Search = Decision in all settings.

Not described: Worst-case hardness for Polynomial-LWE.

†: for a large number of “reasonable” polynomials, up to polynomial factors on noise, assuming some information about the field are known.

1. LWE and Cryptography
   - Regev's encryption scheme
   - Learning With Errors (LWE) and its hardness
2. Ring-based LWE
3. Reductions between Ring-based LWE's
4. Search to Decision
5. Open problems

An encryption scheme [Regev'05]

n "security parameter", q prime, $n \leq m \leq poly(n)$, χ distribution over $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

Alice	Evil	Bruno
$\mathbf{s} \in \mathbb{Z}_q^n$		$\mu \in \{0,1\}$

$$\mathbf{A} \in \mathcal{M}_{m \times n}(\mathbb{Z}_q), \; e_i \hookleftarrow \chi$$

$$\mathbf{b} = \mathbf{A}\,\mathbf{s} + \mathbf{e} \bmod q \quad \longrightarrow \quad (\mathbf{A}, \mathbf{b})$$

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$e' = b' - \langle \mathbf{a}', \mathbf{s}\rangle \bmod q$	$\longleftarrow$ $(\mathbf{a}', b')$	$\leftarrow \mathsf{E}_{\mathbf{A},\mathbf{b}}(\mu) = (\sum_{i\in\mathcal{I}} \mathbf{a}_i, \sum_{i\in\mathcal{I}} b_i + \mu\lfloor \frac{q}{2} \rfloor)$

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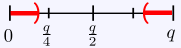
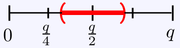
$$\mathsf{Dec}_{\mathbf{s}}(\mathbf{a}', b') = \begin{cases} 0 \text{ if } e' \sim 0 \\ 1 \text{ if } e' \sim \frac{q}{2} \end{cases}$$

Correctness: q, m, χ chosen s.t. $e' = \sum e_i \leq \frac{q}{4}$ whp.

$\mu = 0$



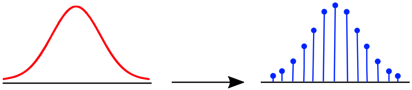
$\mu = 1$



Learning With Errors [R'05]

$n \in \mathbb{N}^*$, $q \leq poly(n)$ a prime
$\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$.

$\chi \to D_r$ **discrete Gaussian distribution**



LWE distribution: Fix $\mathbf{s} \in \mathbb{Z}_q^n$.

$$A_{\mathbf{s},D_r} : \begin{cases} \mathbf{a} \hookleftarrow \mathcal{U}(\mathbb{Z}_q^n) \\ e \hookleftarrow D_r \\ \text{outputs } (\mathbf{a}, b = (\langle \mathbf{a}, \mathbf{s} \rangle + e) \bmod q) \end{cases}$$

Search-LWE$_{q,r}$:

$$\text{From } \Big(m \updownarrow \underbrace{\mathbf{A}}_{n} \;,\; \mathbf{b} = \mathbf{A}\,\mathbf{s} + \mathbf{e} \Big), \text{ find } \mathbf{s}$$

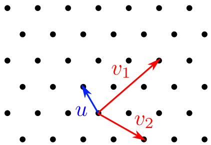
Decision-LWE$_{q,D_r}$: Given $(\mathbf{a}_i, b_i)_{i \le m}$ either from $A_{\mathbf{s},D_r}$ or $\mathcal{U}(\mathbb{Z}_q^n \times \mathbb{Z}_q)$, decide which one was given.

Lattice $\mathcal{L} = \mathbf{A}\mathbb{Z}^n, \lambda_1 =$ length of a shortest vector in $\mathcal{L} \setminus \{0\}$.

ApproxSVP$_\gamma$: Given $d > 0$, decide if $\lambda_1 \le d$ or $\lambda_1 > d\gamma$.

For general lattices:

time	$poly(n)$	$2^{O(n)}$
γ	$2^{\widetilde{O}(n)}$	$poly(n)$



$\|u\| = \lambda_1$

Decision-LWE$_{q,D_r}$**:** Given $(\mathbf{a}_i, b_i)_{i\leq m}$ either from $A_{\mathbf{s},D_r}$ or $\mathcal{U}(\mathbb{Z}_q^n \times \mathbb{Z}_q)$, decide which one was given.

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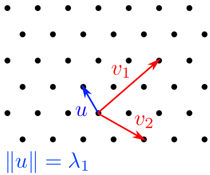

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breaking Regev's encryption $\xRightarrow{classical}$ solving **Decision-LWE** = solving **Search-LWE** $\xRightarrow{quantum}$ solving **ApproxSVP**$_{poly(n)}$

LWE in practice

Perks:

- ✓ simple description, simple operations
- ✓ flexible parameters, many possibilities
- ✓ **post-quantum**

Drawbacks:

- × key-size
- × speed (compared to other)

Frodo† (NIST competitor)	VS	Current crypto RSA 3072-bits	Current crypto ECDH nistp256
Public key $\sim$ **11 KBytes**		~ 400 bytes	32 bytes
Handshake $\sim$ **2.5ms**		~ 5 ms	~ 1.3 ms

†: [BCD++'17]

1 LWE and Cryptography

2 Ring-based LWE
- Polynomial-LWE: ideal lattices
- Ring-LWE: more algebraic number theory

3 Reductions between Ring-based LWE's

4 Search to Decision

5 Open problems

Add structure: ideal lattices

Change $\mathbb{Z} \rightsquigarrow R = \mathbb{Z}[X]/f$
f monic, irreducible, degree n.

Good example: $f = X^n + 1, n = 2^d$.

polynomials

$$s = \sum s_i X^i \in R_q = R/qR$$

Product: $a \cdot s \bmod f$

vectors/matrices

$$\mathbf{s} = (s_0, \ldots, s_{n-1}) \in \mathbb{Z}_q^n$$

Mult. by a = use **Toeplitz matrix**

$$T_f(a) = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ -a_{n-1} & a_0 & \ldots & a_{n-2} \\ \vdots & & \ddots & \vdots \\ -a_1 & -a_2 & \ldots & a_0 \end{bmatrix}$$

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polynomials	**vectors/matrices**
$s = \sum s_i X^i \in R_q = R/qR$	$\mathbf{s} = (s_0, \ldots, s_{n-1}) \in \mathbb{Z}_q^n$
Product: $a \cdot s \bmod f$	Mult. by a = use **Toeplitz matrix**

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Noise: $e = \sum e_i X^i, e_i \hookleftarrow D_{r_i}$.	$\mathbf{e} = (e_0, \ldots, e_{n-1}) \in \mathbb{R}^n$
Sample: $(a, b = a \cdot s + e \bmod qR)$	$(\mathbf{a}, b = T_f(a) \cdot \mathbf{s}^\top + \mathbf{e} \bmod q)$

Classic LWE | **Polynomial-LWE (PLWE)**

$$\underbrace{\mathbf{b}}_{k} = \underbrace{\mathbf{A}}_{k \times n}\,\mathbf{s} + \mathbf{e} \quad \Longrightarrow \quad \underbrace{\mathbf{b}}_{k'n} = \begin{bmatrix} T_f(a_1) \\ \hline T_f(a_2) \\ \hline T_f(a_{k'}) \end{bmatrix}_{n} \mathbf{s} + \begin{bmatrix} \mathbf{e}_1 \\ \hline \mathbf{e}_2 \\ \hline \mathbf{e}_{k'} \end{bmatrix}$$

1 PLWE sample $=$ n correlated LWE samples.

PLWE and its hardness [SSTX'09]

$R = \mathbb{Z}[X]/f$
f monic, irreducible, degree n.

$\vec{r} = \mathsf{diag}(r_i)_{i \leq n}, r_i \geq 0$
$D_{\vec{r}}$ n-dimensional **Gaussian**.

PLWE$_{q,\vec{r},f}$ distribution: Fix $s \in R_q$

$$\mathcal{B}_{s,D_{\vec{r}}} : \begin{cases} a \hookleftarrow \mathcal{U}(R_q) \\ e \hookleftarrow D_{\vec{r}} \\ \text{outputs } (a, b = (a \cdot s + e) \bmod qR) \end{cases}$$

Search-PLWE$_{q,\vec{r},f}$ and **Decision-PLWE$_{q,\vec{r},f}$** defined as before.

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polynomial **ideal:** $aR = \{\text{multiples of } a \text{ in } R\} \longmapsto T_f(a) \cdot \mathbb{Z}^n$: ideal **lattice**

Solve **Search-PLWE** $\Rightarrow$ solve **ApproxSVP$_\gamma$** in **ideal lattices** for $\gamma \leq poly(n)$.

Practice vs. Theory

Perks:

- ✓ fast and compact operations
- ✓ still post-quantum

New Hope†
(NIST competitor)

Public key: ~ 2 KBytes
Handshake: ~ 0.3 ms

Theoretical limitations:

- ✗ γ depends on f's "expansion factor"
- ✗ Working with R relies too much on f

$\rightarrow$ Restricts "good f's"

$\rightarrow$ Difficult proofs, lacks tools and flexibility

†: [ADPS'15]

Number fields and rings

$R = \mathbb{Z}[X]/f$ is a **number ring**. Lives in $K = \mathbb{Q}[X]/f$, a **number field**.

Structure: $K = \mathsf{Span}_{\mathbb{Q}}(1, X, \ldots, X^{n-1})$ where $n = \deg f$

Field embeddings: $\sigma_j(a) = \sum a_i {\alpha_j}^i \in \mathbb{C}$ where $f = \prod_{i \le n}(X - \alpha_j)$.

f has s_1 real roots and $2s_2$ (conjugate) complex roots.

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The space $H = \{(v_1, \ldots, v_n) \in \mathbb{R}^{s_1} \times \mathbb{C}^{2s_2} \,:\, \forall\, i \geq 1, v_{i+s_1+s_2} = \overline{v_{i+s_1}}\}$.

Two representations

Coefficient embedding

$$a \longmapsto \mathbf{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{Q}^n$$

Minkowski embedding

$$a \longmapsto \sigma(a) = (\sigma_1(a), \ldots, \sigma_n(a)) \in H$$

$$\sigma(ab) = (\sigma_i(a)\sigma_i(b))_{i \leq n}$$

The ring of algebraic integers

$$\mathcal{O}_K = \{x \in K \text{ roots of monic polynomials in } \mathbb{Z}[X]\}$$

It is a lattice: $\mathcal{O}_K = \mathbb{Z}b_1 + \ldots + \mathbb{Z}b_n$ for some $b_i \in \mathcal{O}_K$ $(b_i \neq 0)$.

Dual (lattice): $\mathcal{O}_K^\vee = \{\mathbf{y} \in H : \forall\, \mathbf{x} \in \mathcal{O}_K, \langle \mathbf{y}, \mathbf{x} \rangle \in \mathbb{Z}\}$.

- ✓ $\mathcal{O}_K$ is a **regularization** of $R = \mathbb{Z}[X]/f$
  - $R \subsetneq \mathcal{O}_K$ in general
- ✓ $\mathcal{O}_K$ is **intrinsic** to K: its structure **does not depend** on f

It may not be possible to take $1, X, \ldots, X^{n-1}$ as a basis

Computing a $\mathbb{Z}$-basis for $\mathcal{O}_K$ is usually **hard**.

RLWE [LPR'10]

$R \rightsquigarrow \mathcal{O}_K$, **use Minkowski embedding.**
Assume a $\mathbb{Z}$-basis of $\mathcal{O}_K$ is known.

$H = \mathrm{Span}_{\mathbb{R}}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$
$D_{\vec{r}}^H : e_i \hookleftarrow D_{r_i}$, outputs $e = \sum e_i \mathbf{v}_i \in H$.

RLWE$^\vee_{q,\vec{r}}$ distribution: Fix $s \in \mathcal{O}_{K,q}^\vee := \mathcal{O}_K^\vee / q\mathcal{O}_K^\vee$

$$\mathcal{A}^\vee_{s,D_{\vec{r}}} : \begin{cases} a \hookleftarrow \mathcal{U}(\mathcal{O}_{K,q}) \\ e \hookleftarrow D_{\vec{r}}^H \\ \text{outputs } (a, b = (as + e) \bmod q\mathcal{O}_K^\vee) \end{cases}$$

Search-RLWE$^\vee_{q,\vec{r}}$ and **Decision-RLWE$^\vee_{q,\vec{r}}$** defined as before.

"Primal" variant: $s \in \mathcal{O}_{K,q} := \mathcal{O}_K / q\mathcal{O}_K$.

✓ “Canonical” objects

$\downarrow$

✓ Easier proofs/noise management

$\downarrow$

[LPR'10] **Decision-RLWE**$^\vee$ = **Search-RLWE**$^\vee$ for Galois fields

[PRS'17] **Decision** $\Rightarrow$ **ApproxSVP** for **RLWE**$^\vee$**, RLWE, PLWE**

What is left?

- Using **RLWE**$^\vee$ variants $\rightarrow$ Need to deal with $\mathcal{O}_K^\vee$
- $\mathbb{Z}$-basis of $\mathcal{O}_K$? $\rightarrow$ long precomputations for some f's, **non-uniform** reductions
- In practice, f stays cyclotomic. $\rightarrow$ What if cyclotomic fields are “weak”?

(A) Relations between **PLWE, RLWE, RLWE**$^\vee$?

(B) Are **Decision** and **Search** equivalent in Ring-based **LWE**?

(C) Are there "weaker" fields for **ApproxSVP**? For Ring-based **LWE**?

(D) Are there other (better?) structures than ideal lattices for LWE?

Situation and problems

(A) Relations between **PLWE, RLWE, RLWE$^\vee$**?

New Results!

(B) Are **Decision** and **Search** equivalent in Ring-based LWE?

(C) Are there "weaker" fields for **ApproxSVP**? For Ring-based **LWE**?
"Ill-defined": [EHL'14, ELOS'15, CLS'15, HCS'16]

(D) Are there other (better?) structures than ideal lattices for LWE?
Adressed in [LS'15, AD'17, RSSS'17]

1. LWE and Cryptography
2. Ring-based LWE
3. Reductions between Ring-based LWE's
   - Controlled $\mathsf{RLWE}^\vee$ to RLWE
   - From $\mathcal{O}_K$ to R with the conductor
   - Large families of nice polynomials
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Transforming samples [LPR'10, LPR'13]

Goal: map $\mathcal{A}^{\vee}_{s,\Sigma}$ to $\mathcal{A}_{s',\Sigma'}$ and "uniform" to "uniform"

Want: $\theta : \begin{array}{ccc} \mathcal{O}_{K,q} \times \mathcal{O}^{\vee}_{K,q} & \longrightarrow & \mathcal{O}_{K,q} \times \mathcal{O}_{K,q} \\ (a, b) & \longmapsto & (a', b') \end{array}$, respecting the distributions.

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Assume $\exists\ \mathbf{t} \in \mathcal{O}_K$ such that $[\times \mathbf{t}] : \mathcal{O}^{\vee}_{K,q} \simeq \mathcal{O}_{K,q}$. Let $\theta_{\mathbf{t}}(a, b) = (a, \mathbf{t}b \bmod q)$.

If $(a, b) \hookleftarrow \mathcal{A}^{\vee}_{s,\Sigma}$:

$\mathbf{t}b = a(\mathbf{t}s) + \mathbf{t}e$, $\mathbf{t}e \hookleftarrow D^{H}_{\Sigma'}$

$\Sigma' =$ diag $[\,|\sigma_i(\mathbf{t})|\,] \cdot \Sigma \cdot$ diag $[\,|\sigma_i(\mathbf{t})|\,]$

If $(a, b) \hookleftarrow$ uniform:

$[\times \mathbf{t}]$ isomorphism $\Rightarrow (a, \mathbf{t}b)$ uniform

Questions:

1) Does such $\mathbf{t}$ exist? 2) How large is $\mathbf{t}e$?

From RLWE$^\vee$ to RLWE

[LPR'10] Compute $\mathbf{t}$ in $poly(n)$-time with CRT

✓ Existence ✗ Size

$\downarrow$

Our result: An adequate $\mathbf{t}$ with $\|\sigma(\mathbf{t})\| \leq poly(n)$ exists in an adequate lattice.

✓ Existence ✓ Size

Consequence: solving $\textbf{RLWE}_{q,\Sigma'} \Rightarrow$ solving $\textbf{RLWE}^\vee_{q,\Sigma}$

$$\Sigma' \xleftarrow[loss]{poly(n)} \Sigma$$

Ingredients and tools

Our result: An adequate $\mathbf{t}$ with $\|\sigma(\mathbf{t})\| \leq poly(n)$ exists in an adequate lattice.

- **Idea:** use **Gaussian sampling** in $(\mathcal{O}_K^\vee)^{-1}$.
- **Main difficulty:** achieving a small enough standard deviation
  - Require factorization of $q\mathcal{O}_K$ in prime ideals in $\mathcal{O}_K$ (non-uniform reduction)

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- **Main difficulty:** achieving a small enough standard deviation
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- Tools:
  - Inclusion/exclusion
  - Case disjonction on factors' size (norm)
  - "Smoothness parameters" of lattices
  - Tail bounds on Gaussian distributions

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Mapping RLWE to PLWE-like

Goal: map $\mathcal{A}_{s,\Sigma}$ to $\mathcal{B}_{s',\Sigma'}$ and "uniform" to "uniform"

Want: $\theta : \begin{array}{ccc} \mathcal{O}_{K,q} \times \mathcal{O}_{K,q} & \longrightarrow & R_q \times R_q \\ (a, b) & \longmapsto & (a', b') \end{array}$, respecting the distributions.

Result: We can find $[\times \mathbf{t}] : \mathcal{O}_{K,q} \simeq R_q$, such that $\|\sigma(\mathbf{t})\| \leq poly(n)$, for some $\mathbf{t}$ in **the conductor ideal** $\mathcal{C}_R = \{\mathbf{t} \in K \ : \ \mathbf{t}\mathcal{O}_K \subset R\}$.

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$$\begin{tikzcd} R \arrow[rr, leftrightarrow, "?"] \arrow[dr, leftrightarrow, "\checkmark"'] & & \mathcal{O}_K \\ & \mathcal{C}_R \arrow[ur, leftrightarrow, "\checkmark"'] & \end{tikzcd}$$

$\mathcal{C}_R$ "interpolates" between R and $\mathcal{O}_K$

Lemma: if $q \nmid \Delta(f)$, then
$R_q \simeq \mathcal{C}_R / q\mathcal{C}_R \simeq \mathcal{O}_{K,q}$.

- Control $\|\sigma(\mathbf{t})\|$ with the same technique as earlier

"Minkowski noise"

Good candidate: $\theta_{\mathbf{t}}(a, b) = (\mathbf{t}a, \mathbf{t}^2 b \bmod q)$, for $\mathbf{t}$ as above

If $(a, b) \hookleftarrow \mathcal{A}_{s,\Sigma}$:

$\mathbf{t}^2 b = (\mathbf{t}a)(\mathbf{t}s) + \mathbf{t}^2 e$

If $(a, b) \hookleftarrow$ uniform:

$[\times \mathbf{t}]$ isomorphism $\Rightarrow (\mathbf{t}a, \mathbf{t}^2 b)$ uniform

$$e' = \mathbf{t}^2 e \hookleftarrow D^H_{\Sigma_{\mathbf{t}}}, \text{ where } \Sigma_{\mathbf{t}} = \mathrm{diag}[\,|\sigma_i(\mathbf{t})|^2\,] \cdot \Sigma \cdot \mathrm{diag}[\,|\sigma_i(\mathbf{t})|^2\,].$$

e' lives in H, while **PLWE**$_f$ asks for "Coefficient" representation.

“Minkowski” vs “Coefficient”

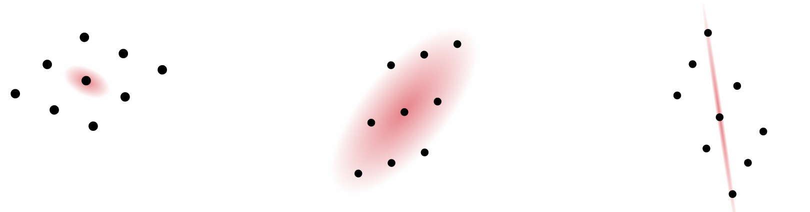
Relation between embeddings:

$$\sigma(a) = \mathbf{V}_f \cdot \mathbf{a}, \text{ with } \mathbf{V}_f = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & & \dots & & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{bmatrix}$$

New noise: $\mathbf{V}_f{}^{-1}\sigma(e') \hookleftarrow D_{\Sigma'}$, with $\Sigma' = \mathbf{V}_f{}^{-\top}\Sigma_{\mathbf{t}}\,\mathbf{V}_f^{-1}$

Possible situations

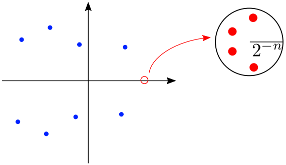
$\mathbf{V}_f^{-1}$ reasonable | $\mathbf{V}_f^{-1}$ too large | $\mathbf{V}_f^{-1}$ too skew



$$\mathbf{V}_f^{-1} = \left(\frac{S_{i,j}}{\Delta_j}\right)_{i,j}, \text{ where } \Delta_j = \prod_{k \neq j}(\alpha_k - \alpha_j).$$

Main difficulties:

- Δ_j can be exponentially small [BM'04]



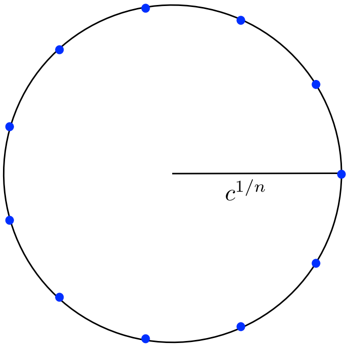
$\min_j |\Delta_j| \leq \tilde{O}(2^{-n})$

- Bound for a large class of polynomials

Goal: A large family of irreducible polynomials in $\mathbb{Z}[X]$ with $\|\mathbf{V}_f^{-1}\| \leq poly(n)$.

(1) $f := X^n - c \in \mathbb{Z}[X]$, with $\alpha_j = c^{1/n}\mathrm{e}^{2i\pi\frac{j}{n}}$.

$\|\mathbf{V}_f^{-1}\|_\infty = 1$.



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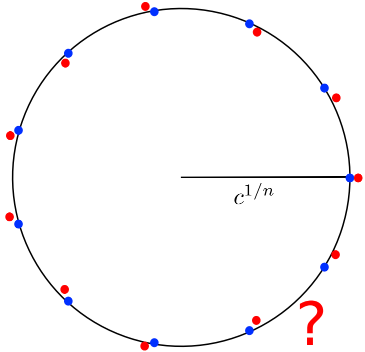
$\|\mathbf{V}_f^{-1}\|_\infty = 1$.

(2) Let $P = \sum_{i=1}^{n/2} p_i X^i \in \mathbb{Z}[X]$.

Perturbation: $g := f + P = \prod_{i=1}^{n}(X - \beta_j)$

If "P small", β_i's should stay close to α_i's.



Perturbations of a good situation

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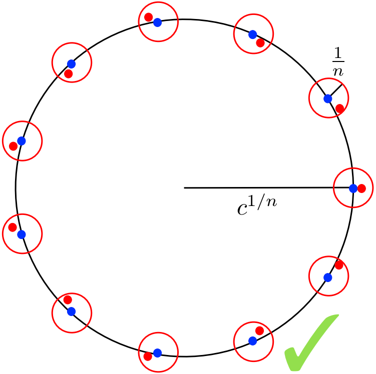

Theorem (Rouché)

If $|P(z)| < |f(z)|$ on a circle, then f and $f+P$ have the same numbers of zeros inside this circle.

Completing the reduction

Result: We can exhibit exponentially many $f \in \mathbb{Z}[X]$, monic and irreducible, such that $\|\mathbf{V}_f^{-1}\| \leq poly(n)$.

For any such f, we have in K_f:

$$\text{solving } \mathbf{PLWE}_{q,\Sigma',f} \Rightarrow \text{solving } \mathbf{RLWE}_{q,\Sigma}$$

$$\begin{tikzcd}
\Sigma' & & \Sigma \arrow[ll, "poly(n)" above, "loss" below] \arrow[dl] \\
& \Sigma_{\mathbf{t}} \arrow[ul] &
\end{tikzcd}$$

Ingredients and tools

Result: We can exhibit exponentially many $f \in \mathbb{Z}[X]$, monic and irreducible, such that $\|\mathbf{V}_f^{-1}\| \leq poly(n)$.

- **Idea:** If β_i's are close to α_i's, then $\|\mathbf{V}_g^{-1}\| \sim \|\mathbf{V}_f^{-1}\|$.
- **Main difficulty:** lower bound on $|\Delta_j| = \prod_{j \neq k} |\beta_k - \beta_j|$.

Ingredients and tools

Result: We can exhibit exponentially many $f \in \mathbb{Z}[X]$, monic and irreducible, such that $\|\mathbf{V}_f^{-1}\| \leq poly(n)$.

- **Idea:** If β_i's are close to α_i's, then $\|\mathbf{V}_g^{-1}\| \sim \|\mathbf{V}_f^{-1}\|$.
- **Main difficulty:** lower bound on $|\Delta_j| = \prod_{j \neq k} |\beta_k - \beta_j|$.
- Steps:
  - Bound $|P(z)|, |f(z)|$ on $D(\alpha_i, \frac{1}{n}) \Rightarrow$ conditions on $c, \|P\|_1$.
  - Assume conditions are met.
    Rouché's theorem implies $|\Delta_j| \geq \prod \left(\underbrace{|\alpha_k - \alpha_j|}_{\text{well-known}} - \frac{2}{n} \right)$
  - Irreducibility when c is a large enough prime

Given: $\left(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \right)$ + disting. oracle, find $\mathbf{s}$.

Main steps:

Generate **RLWE**-like samples using Gaussians $t_i \hookleftarrow D_{\sigma,\mathcal{O}_K}$	$\longrightarrow$	Get good approximations of noise in poly time [PRS'17]

Difficulty: Find minimal σ s.t. linear combinations of t_i's look uniform.

Result: Leftover Hash Lemma over number rings.
$a_1, \ldots, a_m$: rows of $\mathbf{A}$. Standard dev. $\sigma \geq \widetilde{O}(\sqrt{n} \cdot \Delta_K^{1/n} \cdot q^{1/m})$.

If $t_i \hookleftarrow D_{\sigma,\mathcal{O}_K}$, then $\sum_{i \leq m} a_i t_i$ is statistically indistinguishable from uniform.

1 LWE and Cryptography

2 Ring-based LWE

3 Reductions between Ring-based LWE's

4 Search to Decision

5 Open problems

Open problems related to this work

(A) Make reductions uniform.

(B) $\|V_f^{-1}\| \leq \widetilde{O}(n^{3.5})$ in proof vs. $\|V_f^{-1}\| \sim 1$ in practice. Improvement?

(C) Are there "weaker" fields for **ApproxSVP**? For Ring-based **LWE**?

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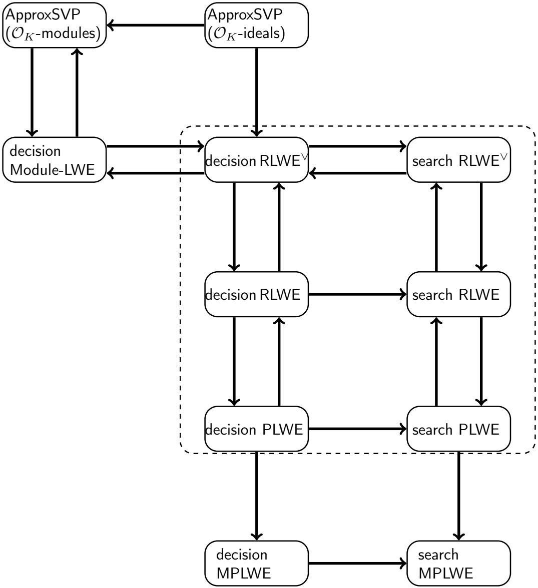
ApproxSVP ($\mathcal{O}_K$-modules)
ApproxSVP ($\mathcal{O}_K$-ideals)
decision Module-LWE
decision RLWE$^\vee$
search RLWE$^\vee$
decision RLWE
search RLWE
decision PLWE
search PLWE
decision MPLWE
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Thank you :)